Two-body problem in Scalar-Tensor theories, an Effective-One-Body approach

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Recent Developments in General Relativity

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arXiv:1703.05360 FLJ - Nathalie Deruelle

• GW150914 : The very first observation of a BBH coalescence by LIGO-Virgo has opened **a new era in gravitational wave astronomy.**

• Opportunity to bring **new tests of modified gravities**, in the strong-field regime near merger, a topic which is for the moment still in infancy.

Motivations

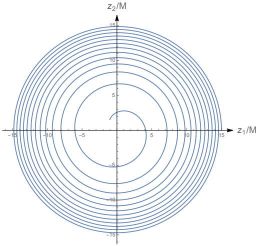
In General Relativity, "Effective-One-Body" (EOB) :

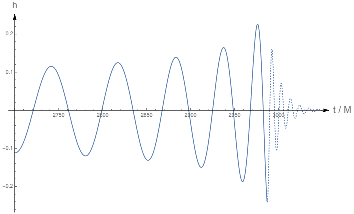
• Map the two-body PN dynamics to the motion of a **test particle** in an **effective SSS metric** [Buonanno-Damour 98]

$$H(Q,P)\ ,\quad \epsilon = \left(\frac{v}{c}\right)^2 \qquad \longrightarrow \qquad H_e(q,p)\ ,\quad ds_e^2 = g^e_{\mu\nu}dx^\mu dx^\nu$$

$$\boxed{H_e = f_{\mathrm{EOB}}(H)}$$

• Defines a resummation of the PN dynamics, hence describes **analytically** the coalescence of 2 compact objects in **General Relativity**, from inspiral **to merger**.





• Instrumental to build libraries of waveform templates for LIGO-VIRGO

Motivations

Our proposition [arXiv:1703.05360]

- Can we extend the EOB approach to modified gravities ?
- Consider the simplest and most studied example of **massless Scalar-Tensor theories**.

• First buiding block : map the conservative part of the two-body dynamics onto the geodesic of an effective metric.

• ST-extension of [Buonanno-Damour 98]

We adopt the conventions of Damour and Esposito-Farèse [DEF 92, 95]

ST action in the Einstein-frame ($G_* \equiv c \equiv 1$)

$$S_{EF} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) + S_m \left[\Psi, \mathcal{A}^2(\varphi) g_{\mu\nu} \right]$$

• Einstein metric $g_{\mu\nu}$ free dynamics : Einstein-Hilbert term ; ordinary kinematical term for φ ;

• **BUT** matter Ψ is minimally coupled to the Jordan metric $\tilde{g}_{\mu\nu}$:

$$\boxed{\tilde{g}_{\mu\nu} \equiv \mathcal{A}^2(\varphi) g_{\mu\nu}}$$

where $\mathcal{A}(\varphi)$ **defines** the ST theory (GR : $\mathcal{A}(\varphi) = cst$).

• Encompass the Einstein Equivalence Principle what about S_m ?

N-body problem in Scalar-Tensor theories

Phenomenological approach : Skeletonize extended bodies as point particles

- Non-self-gravitating case

$$S_m = -\sum_A \int d\lambda \sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,\tilde{m}_A$$

i.e. particles follow **geodesics** of $\tilde{g}_{\mu\nu}$ (WEP).

- When self-gravity is not negligible (neutron stars, black holes),

$$S_m = -\sum_A \int d\lambda \sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,\tilde{m}_A(\varphi)$$

$\tilde{m}_A(\varphi)$ is a function of the local value of φ to encompass the effect of the background scalar field on the equilibrium of a body. [Eardley 75, DEF 92]
$\tilde{m}_A(\varphi)$ depends on the theory $\mathcal{A}(\varphi)$ and on the EOS of body A.

$\rightarrow$ SEP violation

Matter action in the Jordan-frame

$$S_m = -\sum_A \int d\lambda \sqrt{-\tilde{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, \tilde{m}_A(\varphi)$$

Since $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi) g_{\mu\nu}$:

Matter action in the Einstein-frame

$$S_m = -\sum_A \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, m_A(\varphi)$$

where we have defined the Einstein-frame mass :

$$\boxed{m_A(\varphi) \equiv \mathcal{A}(\varphi) \tilde{m}_A(\varphi)}$$

The two-body problem hence depends only on 2 fundamental functions, $m_A(\varphi)$ and $m_B(\varphi)$ that encompass **completely** the theory and body-dependence. (GR is recovered when $m_A = cst$, $m_B = cst$.)

I) THE TWO – BODY HAMILTONIAN AT 2PK ORDER

The two-body Lagrangian

Our starting point : what is known today

Two-body Scalar-Tensor Lagrangian [DEF 93][Mirshekari, Will 13]

- conservative 2PK dynamics : $\mathcal{O}((\frac{v}{c})^4) \sim \mathcal{O}((\frac{m}{r})^2)$ corrections to Kepler
- Weak field expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$$
$$\varphi = \varphi_0 + \delta\varphi$$

- Harmonic coordinates $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$

Generalizes the 2PN GR Lagrangian [Damour, Deruelle 82]

(note that GR dynamics is known at 4PN today)

- the fundamental functions $m_A(\varphi)$ and $m_B(\varphi)$ are expanded around φ_0 :

$$\ln m_A(\varphi) \equiv \ln m_A^0 + \alpha_A^0(\varphi - \varphi_0) + \beta_A^0(\varphi - \varphi_0)^2 + {\beta'}_A^0(\varphi - \varphi_0)^3 + \cdots$$
$$\ln m_B(\varphi) \equiv \ln m_B^0 + \alpha_B^0(\varphi - \varphi_0) + \beta_B^0(\varphi - \varphi_0)^2 + {\beta'}_B^0(\varphi - \varphi_0)^3 + \cdots$$

i.e. the 2PK Lagrangian depends on 8 fundamental **parameters**.

The Jordan-frame Mirshekari-Will Lagrangian is to be translated in terms of the Einstein-frame parametrization introduced above

Two-body 2PK Lagrangian

$$L = -m_A^0 - m_B^0 + L_{\rm K} + L_{\rm 1PK} + L_{\rm 2PK} + \cdots$$

$$\vec{N} \equiv \frac{\vec{Z}_A - \vec{Z}_B}{R}\,, \quad \vec{V}_A \equiv \frac{d\vec{Z}_A}{dt}\,, \quad R \equiv \mid \vec{Z}_A - \vec{Z}_B \mid\,, \quad \vec{A}_A \equiv \frac{d\vec{V}_A}{dt}$$

- Keplerian order :

$$\boxed{L_{\rm K} = \frac{1}{2} m_A^0 V_A^2 + \frac{1}{2} m_B^0 V_B^2 + \frac{G_{AB} m_A^0 m_B^0}{R}} \qquad \text{where} \qquad G_{AB} \equiv 1 + \alpha_A^0 \alpha_B^0$$

- post-Keplerian (1PK) :

$$\begin{aligned} L_{\rm 1PK} = & \frac{1}{8} m_A^0 V_A^4 + \frac{1}{8} m_B^0 V_B^4 \\ & + \frac{G_{AB} m_A^0 m_B^0}{R} \left(\frac{3}{2}(V_A^2 + V_B^2) - \frac{7}{2} \vec{V}_A \cdot \vec{V}_B - \frac{1}{2} (\vec{N} \cdot \vec{V}_A)(\vec{N} \cdot \vec{V}_B) + \bar{\gamma}_{AB} (\vec{V}_A - \vec{V}_B)^2 \right) \\ & - \frac{G_{AB}^2 m_A^0 m_B^0}{2R^2} \left(m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A) \right) \end{aligned}$$

$$\text{where} \quad \bar{\gamma}_{AB} \equiv -\frac{2\alpha_A^0 \alpha_B^0}{1 + \alpha_A^0 \alpha_B^0} \qquad \bar{\beta}_A \equiv \frac{1}{2} \frac{\beta_A^0 (\alpha_B^0)^2}{(1 + \alpha_A^0 \alpha_B^0)^2} \ (A \leftrightarrow B)$$

- post-post-Keplerian (2PK) :

$$
\begin{aligned}
L_{\rm 2PK} &= \frac{1}{16} m_A^0 V_A^6 \\
&+ \frac{G_{AB} m_A^0 m_B^0}{R} \Bigg[\frac{1}{8}(7+4\bar\gamma_{AB}) \left(V_A^4 - V_A^2 (\vec N\cdot\vec V_B)^2 \right) - (2+\bar\gamma_{AB}) V_A^2 (\vec V_A\cdot\vec V_B) + \frac{1}{8}(\vec V_A\cdot\vec V_B)^2 \\
&\qquad + \frac{1}{16}(15+8\bar\gamma_{AB}) V_A^2 V_B^2 + \frac{3}{16}(\vec N\cdot\vec V_A)^2(\vec N\cdot\vec V_B)^2 + \frac{1}{4}(3+2\bar\gamma_{AB})\vec V_A\cdot\vec V_B(\vec N\cdot\vec V_A)(\vec N\cdot\vec V_B) \Bigg] \\
&+ \frac{G_{AB}^2 m_B^0 (m_A^0)^2}{R^2} \Bigg[\frac{1}{8}\left(2+12\bar\gamma_{AB}+7\bar\gamma_{AB}^2+8\bar\beta_B-4\delta_A\right) V_A^2 + \frac{1}{8}\left(14+20\bar\gamma_{AB}+7\bar\gamma_{AB}^2+4\bar\beta_B-4\delta_A\right) V_B^2 \\
&\qquad - \frac{1}{4}\left(7+16\bar\gamma_{AB}+7\bar\gamma_{AB}^2+4\bar\beta_B-4\delta_A\right)\vec V_A\cdot\vec V_B - \frac{1}{4}\left(14+12\bar\gamma_{AB}+\bar\gamma_{AB}^2-8\bar\beta_B+4\delta_A\right)(\vec V_A\cdot\vec N)(\vec V_B\cdot\vec N) \\
&\qquad + \frac{1}{8}\left(28+20\bar\gamma_{AB}+\bar\gamma_{AB}^2-8\bar\beta_B+4\delta_A\right)(\vec N\cdot\vec V_A)^2 + \frac{1}{8}\left(4+4\bar\gamma_{AB}+\bar\gamma_{AB}^2+4\delta_A\right)(\vec N\cdot\vec V_B)^2 \Bigg] \\
&+ \frac{G_{AB}^3 (m_A^0)^3 m_B^0}{2R^3}\left[1+\frac{2}{3}\bar\gamma_{AB}+\frac{1}{6}\bar\gamma_{AB}^2+2\bar\beta_B+\frac{2}{3}\delta_A+\frac{1}{3}\epsilon_B\right] + \frac{G_{AB}^3 (m_A^0)^2 (m_B^0)^2}{8R^3}\left[19+8\bar\gamma_{AB}+8(\bar\beta_A+\bar\beta_B)+4\zeta\right] \\
&- \frac{1}{8} G_{AB} m_A^0 m_B^0 \left(2(7+4\bar\gamma_{AB}) \vec A_A\cdot\vec V_B(\vec N\cdot\vec V_B) + \vec N\cdot\vec A_A(\vec N\cdot\vec V_B)^2 - (7+4\bar\gamma_{AB})\vec N\cdot\vec A_A V_B^2 \right) \\
&\qquad + (A\leftrightarrow B)
\end{aligned}
$$

where $\delta_A \equiv \frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$ $\quad \epsilon_A \equiv \frac{(\beta'_A\alpha_B^3)^0}{(1+\alpha_A^0\alpha_B^0)^3}$ $\quad \zeta \equiv \frac{\beta_A^0\alpha_A^0\alpha_B^0\beta_B^0}{(1+\alpha_A^0\alpha_B^0)^3}$ $(A\leftrightarrow B)$

L is written in Harmonic coordinates, and depends on R, $\vec{V}_A$ and on the accelerations $\vec{A}_A$ at 2PK level :

$$L_{\rm 2PK} \ni \boxed{-\frac{1}{8}G_{AB}m_A^0 m_B^0\left(2(7+4\bar{\gamma}_{AB})\vec{A}_A\cdot\vec{V}_B(\vec{N}\cdot\vec{V}_B)+\vec{N}\cdot\vec{A}_A(\vec{N}\cdot\vec{V}_B)^2-(7+4\bar{\gamma}_{AB})\vec{N}\cdot\vec{A}_A V_B^2\right)}$$

$\rightarrow$ **Order reduction ?**

contact transformation [Schäfer 83, Damour-Schäfer 91]

Order reduction : contact transformation [Schäfer 83, Damour-Schäfer 91]

1) Add a generic 2PK total time derivative,

$$\boxed{L \to L + \frac{df}{dt} \equiv L_f}$$

$$\begin{aligned}\frac{f}{m_A^0 m_B^0} \equiv G_{AB}\Big[&(f_1 V_A^2 + f_2 \vec{V}_A\cdot\vec{V}_B + f_3 V_B^2)(\vec{N}\cdot\vec{V}_A) - (f_4 V_A^2 + f_5 \vec{V}_A\cdot\vec{V}_B + f_6 V_B^2)(\vec{N}\cdot V_B)\\ &+ f_7(\vec{N}\cdot\vec{V}_A)^3 + f_8(\vec{N}\cdot\vec{V}_A)^2(\vec{N}\cdot\vec{V}_B) - f_9(\vec{N}\cdot\vec{V}_B)^2(\vec{N}\cdot\vec{V}_A) - f_{10}(\vec{N}\cdot\vec{V}_B)^3\Big]\\ &+ G_{AB}^2\left[f_{11}\left(\frac{m_A^0}{R}\right)(\vec{N}\cdot\vec{V}_A) + f_{12}\left(\frac{m_B^0}{R}\right)(\vec{N}\cdot\vec{V}_A) - f_{13}\left(\frac{m_A^0}{R}\right)(\vec{N}\cdot\vec{V}_B) - f_{14}\left(\frac{m_B^0}{R}\right)(\vec{N}\cdot\vec{V}_B)\right]\end{aligned}$$

where f is a generic function, depending on 14 parameters f_i.

2) Replace the accelerations by their leading on-shell expressions :

$$L_f \rightarrow L_f\left(\vec{A}_A \rightarrow -\vec{N}\frac{G_{AB}m_B^0}{R^2}\ ,\ \vec{A}_B \rightarrow \vec{N}\frac{G_{AB}m_A^0}{R^2}\right) \equiv L_f^{red}$$

⇔ implicit coordinate change (contact transformation) : $\vec{Z}_A \rightarrow \vec{Z}_A + \delta\vec{Z}_A$

$$\begin{aligned}\delta\vec{Z}_A =& \frac{G_{AB}m_B^0}{8}\left[2(7+4\bar{\gamma}_{AB})\vec{V}_B(\vec{N}\cdot\vec{V}_B) - \vec{N}\left((7+4\bar{\gamma}_{AB})V_B^2 - (\vec{N}\cdot\vec{V}_B)^2\right)\right] \\ & - G_{AB}m_B^0\left[\vec{V}_A\left(2f_1(N\cdot V_A) - 2f_4(N\cdot V_B)\right) + \vec{V}_B\left(f_2(N\cdot V_A) - f_5(N\cdot V_B)\right)\right. \\ & \left. +\vec{N}\left(f_1V_A^2 + f_2 V_A\cdot V_B + f_3 V_B^2 + 3f_7(N\cdot V_A)^2 + 2f_8(N\cdot V_A)(N\cdot V_B) - f_9(N\cdot V_B)^2 + f_{11}\frac{G_{AB}m_A^0}{R} + f_{12}\frac{G_{AB}m_B^0}{R}\right)\right]\end{aligned}$$

- We have on hand a **whole class of coordinate systems** labeled by 14 parameters f_i for which L_f^{red} is **ordinary**.
- The harmonic coordinates do not belong to this class.

The centre-of-mass two-body 2PK Hamiltonians

- The two-body Hamiltonians are derived from L_f^{red} through a further **Legendre transformation** :

$$\vec{P}_A = \frac{\partial L_f^{red}}{\partial \vec{V}_A} \, , \quad \vec{P}_B = \frac{\partial L_f^{red}}{\partial \vec{V}_B} \, , \quad H = \vec{P}_A \cdot \vec{V}_A + \vec{P}_B \cdot \vec{V}_B - L_f^{red}$$

- In the centre-of-mass frame : $\boxed{\vec{P}_A + \vec{P}_B \equiv \vec{0}}$

  i.e. $\vec{Z} \equiv \vec{Z}_A - \vec{Z}_B$ and $\vec{P} \equiv \vec{P}_A = -\vec{P}_B$

- The relative motion is planar $\rightarrow$ use polar coodinates $(Q, P) \equiv (R, \Phi, P_R, P_\Phi)$

The centre-of-mass two-body 2PK Hamiltonians

General structure of a centre-of-mass frame Hamiltonian $H(Q,P)$

17 coefficients

$$H = M + \left(\frac{P^2}{2\mu} - \mu\frac{G_{AB}M}{R}\right) + H^{1\text{PK}} + H^{2\text{PK}} + \cdots$$

- $\dfrac{H^{1\text{PK}}}{\mu} = \left(h_1^{1\text{PK}}\hat{P}^4 + h_2^{1\text{PK}}\hat{P}^2\hat{P}_R^2 + h_3^{1\text{PK}}\hat{P}_R^4\right) + \dfrac{1}{\hat{R}}\left(h_4^{1\text{PK}}\hat{P}^2 + h_5^{1\text{PK}}\hat{P}_R^2\right) + \dfrac{h_6^{1\text{PK}}}{\hat{R}^2}$
- $\dfrac{H^{2\text{PK}}}{\mu} = \left(h_1^{2\text{PK}}\hat{P}^6 + h_2^{2\text{PK}}\hat{P}^4\hat{P}_R^2 + h_3^{2\text{PK}}\hat{P}^2\hat{P}_R^4 + h_4^{2\text{PK}}\hat{P}_R^6\right)$

$$+ \frac{1}{\hat{R}}\left(h_5^{2\text{PK}}\hat{P}^4 + h_6^{2\text{PK}}\hat{P}_R^2\hat{P}^2 + h_7^{2\text{PK}}\hat{P}_R^4\right) + \frac{1}{\hat{R}^2}\left(h_8^{2\text{PK}}\hat{P}^2 + h_9^{2\text{PK}}\hat{P}_R^2\right) + \frac{h_{10}^{2\text{PK}}}{\hat{R}^3}$$

$$\mu \equiv \frac{m_A^0 m_B^0}{M}\,, \qquad M \equiv m_A^0 + m_B^0$$

10+6+1=17

In the Scalar-Tensor case :

The 17 h_i^{NPK} coefficients are computed explicitly and depend on :

- the 14 f_i (coordinate system) parameters
- the 8 fundamental parameters built from $m_A(\varphi)$ and $m_B(\varphi)$

Reminder :

$$\ln m_A(\varphi) \equiv \ln m_A^0 + \alpha_A^0(\varphi - \varphi_0) + \beta_A^0(\varphi - \varphi_0)^2 + {\beta'}_A^0(\varphi - \varphi_0)^3 + \cdots$$
$$\ln m_B(\varphi) \equiv \ln m_B^0 + \alpha_B^0(\varphi - \varphi_0) + \beta_B^0(\varphi - \varphi_0)^2 + {\beta'}_B^0(\varphi - \varphi_0)^3 + \cdots$$

Recap

- Start from the 2PK two-body Lagrangian (8 parameters)
- Order reduce it through a contact transformation (14 f_i parameters)
- Deduce the 17 h_i^{NPK} coefficients of $H(Q, P)$

$H(Q, P)$ contains all the information concerning the two-body dynamics at 2PK order, but is **heavy** !

We will contrast it with a much simpler problem, the **geodesic of a test particle in an effective metric**.

II) A TEST PARTICLE IN A SSS EFFECTIVE METRIC

Geodesic motion in a static, spherically symmetric metric

In Schwarzschild-Droste coordinates (equatorial plane $\theta = \pi/2$) :

$$\boxed{ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2}$$

$A(r)$ and $B(r)$ are arbitrary.

- by staticity and spherical symmetry

$$u_t = -A\frac{dt}{d\lambda} \equiv -E \ , \quad u_\phi = r^2\frac{d\phi}{d\lambda} \equiv L$$

- 4-velocity normalization

$$u^\mu u_\mu \equiv -1$$

Combining the 3 constants of motion :

Radial EOM ($u \equiv 1/r$)

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{AB}F(u) \quad \text{with} \quad F(u) \equiv E^2 - A(u)\left(1 + L^2 u^2\right)$$

- Circular orbits : $F(u) = 0$ and $F'(u) = 0$, hence

$$\boxed{L^2(u) = -\frac{A'}{(Au^2)'} \;, \quad E(u) = A\sqrt{\frac{2u}{(Au^2)'}}}$$

- Easily yields predictions in the Strong-field regime, e.g. ISCO :

$$F'(u_{\mathrm{ISCO}}) = F''(u_{\mathrm{ISCO}}) = 0 \quad \Rightarrow \quad \boxed{\frac{A''}{A'} = \frac{(Au^2)''}{(Au^2)'}}$$

- One hence easily computes the ISCO orbital frequency

$$\boxed{\omega_{\mathrm{ISCO}} = \frac{d\phi}{dt} = \frac{L}{E} Au^2 \bigg|_{u_{\mathrm{ISCO}}}}$$

Note : In these Schwarzschild-Droste coordinates, the dynamics depends only on $A = -g^e_{00}$ for circular orbits

The effective Hamiltonian H_e

Equivalently, this dynamics is described by the **Lagrangian**

$$L_e = -\mu\sqrt{-g^e_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -\mu\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2}$$

Effective Hamiltonian $H_e(q,p)$:

$$H_e(q,p) = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{\hat{r}^2}\right)} \quad \text{with} \quad p_r \equiv \frac{\partial L_e}{\partial \dot{r}} \quad , \quad p_\phi \equiv \frac{\partial L_e}{\partial \dot{\phi}}$$

Can be **expanded** :

$$\boxed{\begin{aligned} A(r) &= 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots \\ B(r) &= 1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots \end{aligned}}$$

i.e. depend on **5 effective parameters** at 2PK order, to be determined.

Recap

• On one side, the 2PK two-body Hamiltonian $H(Q, P)$, depending on 17 parameters h_i^{NPK}

• On the other side, a simple effective Hamiltonian $H_e(q, p)$, depending on 5 parameters a_i, b_i.

Can we build a map between both Hamiltonians ?

EOB mapping : [Buonanno, Damour 98]

Requires imposing a functional relation $H_e = f_{\mathrm{EOB}}(H)$ by means of a canonical transformation

III) THE EOB MAPPING

1) Exploit the power of canonical transformations :

$$H(Q,P) \to H(q,p)$$

We take as a generic ansatz $G(Q,p)$ that depends on 9 parameters at 2PK order :

$$G(Q,p) = R\,p_r \left[\left(\alpha_1 \mathcal{P}^2 + \beta_1 \hat{p}_r^2 + \frac{\gamma_1}{\hat{R}} \right) + \left(\alpha_2 \mathcal{P}^4 + \beta_2 \mathcal{P}^2 \hat{p}_r^2 + \gamma_2 \hat{p}_r^4 + \delta_2 \frac{\mathcal{P}^2}{\hat{R}} + \epsilon_2 \frac{\hat{p}_r^2}{\hat{R}} + \frac{\eta_2}{\hat{R}^2} \right) + \cdots \right]$$

$$\mu = \frac{m_A^0 m_B^0}{m_A^0 + m_B^0} \; , \quad M = m_A^0 + m_B^0$$

$$r(Q,p) = R + \frac{\partial G}{\partial p_r} \; , \quad \phi(Q,p) = \Phi + \frac{\partial G}{\partial p_\phi} \; , \quad P_R(Q,p) = p_r + \frac{\partial G}{\partial R} \; , \quad P_\Phi(Q,p) = p_\phi + \frac{\partial G}{\partial \Phi}$$

- does not depend on time (conservative), nor on Φ (isotropic)
- generates 1PK and higher order coordinate changes

2) Relate H to H_e through a functional relation $H_e = f_{\rm EOB}(H)$

- at 2PK :

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\bar{\nu}_1}{2}\left(\frac{H(q,p)-M}{\mu}\right) + \bar{\nu}_2\left(\frac{H(q,p)-M}{\mu}\right)^2 + \cdots\right]$$

the Hamiltonians identifying at Keplerian order.

- In GR, $\bar{\nu}_1 = \nu$ while $\bar{\nu}_2 \cdots = 0$ at least up to 4PN

The exact quadratic relation

As proven recently to all orders from PM [Damour 2016]:

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

where $\nu = \frac{m_A^0 m_B^0}{(m_A^0 + m_B^0)^2}$, $M = m_A^0 + m_B^0$, $\mu = \frac{m_A^0 m_B^0}{M}$

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

- H_e depends on 5 parameters

$$A(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots \ , \quad B(r) = 1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots$$

- H depends on 17 coefficients (h_i^{NPK}) ;
- The canonical transformation depends on 9 parameters $(\alpha_i,\ \beta_i, ...)$;

$$\boxed{17 = 9 + 5 + 3}$$

Hence, 3 constraints on the h_i^{NPK} coefficients of the two-body Hamiltonian.

$\rightarrow$ **The two-body problem can be mapped towards a geodesic only for a subclass of theories.**

- **At 1PK order, one constraint :**

$$\boxed{2h_2^{\rm 1PK} + 3h_3^{\rm 1PK} = 0}$$

By Lorentz invariance of the kinematical terms $m_A^0\sqrt{1 - V_A^2}$ at the Lagrangian level , $h_2^{\rm 1PK} = h_3^{\rm 1PK} = 0$ in ST theory.

- **At 2PK order, two constraints :** the first one

$$\boxed{h_4^{\rm 2PK} = -\frac{2}{45}\left(12h_2^{\rm 2PK} + 18h_3^{\rm 2PK} + (h_2^{\rm 1PK})^2\right)}$$

is no more restrictive.

However, the second one

$$h_1^{\rm 2PK} + \frac{7}{3} h_2^{\rm 2PK} + h_3^{\rm 2PK} + h_5^{\rm 2PK} + h_6^{\rm 2PK} + h_7^{\rm 2PK} =$$

$$- \frac{h^{\rm K}}{128}(5 + 2\nu + 5\nu^2) + \frac{1}{8}(1+\nu)\left((3h_1^{\rm 1PK} + h_2^{\rm 1PK})h^{\rm K} + h_4^{\rm 1PK} + h_5^{\rm 1PK}\right) + \frac{5}{2} h_1^{\rm 1PK}\left(7h_1^{\rm 1PK} h^{\rm K} + 2(h_4^{\rm 1PK} + h_5^{\rm 1PK})\right)$$

$$+ \frac{1}{6} h_2^{\rm 1PK}\left(13 h_2^{\rm 1PK} h^{\rm K} + 10(h_4^{\rm 1PK} + h_5^{\rm 1PK})\right) + \frac{35}{3} h_1^{\rm 1PK} h_2^{\rm 1PK} h^{\rm K} \ ,$$

is restrictive.

- satisfied by the Scalar-Tensor coefficients, for any f_i...
- but not by Electrodynamics.

In Scalar-Tensor theories, injecting the expressions of h_i^{NPK}, the identification

$$\frac{H_e(q,p)}{\mu}-1=\left(\frac{H(q,p)-M}{\mu}\right)\left[1+\frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

• yields a **unique** solution for H_e that does not depend on the f_i parameters (covariance).

• H_e contain all the 2PK physical information contained in $H(Q,P)$.

Part of the **complexity** of $H(Q,P)$, e.g. f, is **hidden in the canonical transformation** :

$$G(Q,p) = R\,p_r\left(\alpha_1\mathcal{P}^2 + \beta_1\hat{p}_r^2 + \frac{\gamma_1}{\hat{R}} + \alpha_2\mathcal{P}^4 + \beta_2\mathcal{P}^2\hat{p}_r^2 + \gamma_2\hat{p}_r^4 + \delta_2\frac{\mathcal{P}^2}{\hat{R}} + \epsilon_2\frac{\hat{p}_r^2}{\hat{R}} + \frac{\eta_2}{\hat{R}^2} + \cdots\right)$$

whose 9 parameters read :

$$\alpha_1 = -\frac{\nu}{2}\ ,\quad \beta_1 = 0\ ,\quad \gamma_1 = G_{AB}\left(1+\bar{\gamma}_{AB}+\frac{\nu}{2}\right)\ ,\quad \alpha_2 = \frac{1}{8}(1-\nu)\nu\ ,\quad \beta_2 = 0\ ,\quad \gamma_2 = \frac{\nu^2}{2}\ ,$$

$$\delta_2 = G_{AB}\left[f_6\frac{m_A^0}{M} + f_1\frac{m_B^0}{M} - \nu\left(f_1+f_6+(-f_3+f_5+f_6)\frac{m_A^0}{M} + (f_1+f_2-f_4)\frac{m_B^0}{M} - \frac{3}{2}(1+\bar{\gamma}_{AB}) + \frac{\nu}{8}\right)\right]\ ,$$

$$\epsilon_2 = G_{AB}\left[-\frac{\nu^2}{8} + f_{10}\frac{m_A^0}{M} + f_7\frac{m_B^0}{M} - \nu\left(f_7+f_{10}+(f_9+f_{10})\frac{m_A^0}{M} + (f_7+f_8)\frac{m_B^0}{M}\right)\right]\ ,$$

$$\eta_2 = \frac{1}{8}G_{AB}^2\left[8\langle\bar{\beta}\rangle - 4\langle\delta\rangle + 4\bar{\gamma}_{AB} + 3\bar{\gamma}_{AB}^2 + \nu\left(-38 + 4(\bar{\beta}_A+\bar{\beta}_B) - 24\bar{\gamma}_{AB} + 2\nu\right)\right]$$
$$+ G_{AB}^2\left(f_{13}\frac{m_A^0}{M} + f_{12}\frac{m_B^0}{M} + \nu(f_{11}-f_{12}-f_{13}+f_{14})\right)$$

The Scalar-Tensor effective metric

$$ds_e^2 = -A(r)dt + B(r)dr^2 + r^2 d\theta^2$$

Scalar-Tensor effective metric

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \left[2\nu + \delta a_3^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^3 + \cdots$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \left[2(2-3\nu) + \delta b_2^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots$$

Consistent with General Relativity : $m_A(\varphi) = cst$ yields back

General Relativity 2PN effective metric [Buonanno, Damour 98]

$$A_{\mathrm{GR}}(r) = 1 - 2\left(\frac{G_* M}{r}\right) + 2\nu\left(\frac{G_* M}{r}\right)^3 + \cdots$$

$$B_{\mathrm{GR}}(r) = 1 + 2\left(\frac{G_* M}{r}\right) + 2(2-3\nu)\left(\frac{G_* M}{r}\right)^2 + \cdots$$

(*i*) The "bare" gravitational constant G_* is replaced by the effective one

$$\boxed{G_* \to G_{AB} \equiv 1 + \alpha_A^0 \alpha_B^0}$$

at all orders.

(*ii*) At 1PK level,

$$A(r) = 1 - 2\left(\frac{G_{AB} M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB} M}{r}\right)^2 + \cdots$$
$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB} M}{r}\right) + \cdots$$

one recognizes the **PPN Eddington metric** written in Droste coordinates, with :

$$\boxed{\beta^{\mathrm{Edd}} = 1 + \langle\bar{\beta}\rangle \ , \quad \gamma^{\mathrm{Edd}} = 1 + \bar{\gamma}_{AB}}$$

Where

$$\langle\bar{\beta}\rangle \equiv \frac{m_A^0 \bar{\beta}_B + m_B^0 \bar{\beta}_A^0}{m_A^0 + m_B^0} \qquad \bar{\gamma}_{AB} \equiv -\frac{2\alpha_A^0 \alpha_B^0}{1 + \alpha_A^0 \alpha_B^0} \qquad \bar{\beta}_A \equiv \frac{1}{2}\frac{\beta_A^0 (\alpha_B^0)^2}{(1 + \alpha_A^0 \alpha_B^0)^2}$$

Scalar-Tensor effective metric

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \left[2\nu + \delta a_3^{\rm ST}\right]\left(\frac{G_{AB}M}{r}\right)^3 + \cdots$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \left[2(2-3\nu) + \delta b_2^{\rm ST}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots$$

(iii) 2PK corrections

$$\delta a_3^{\rm ST} \equiv \frac{1}{12}\Big[-20\bar{\gamma}_{AB} - 35\bar{\gamma}_{AB}^2 - 24\langle\bar{\beta}\rangle(1 - 2\bar{\gamma}_{AB}) + 4\big(\langle\delta\rangle - \langle\epsilon\rangle\big) + \nu\Big(-36(\bar{\beta}_A + \bar{\beta}_B) + 4\bar{\gamma}_{AB}(10 + \bar{\gamma}_{AB}) + 4(\epsilon_A + \epsilon_B) + 8(\delta_A + \delta_B) - 24\zeta\Big)\Big]$$

$$\delta b_2^{\rm ST} \equiv \Big[4\langle\bar{\beta}\rangle - \langle\delta\rangle + \bar{\gamma}_{AB}(9 + \frac{19}{4}\bar{\gamma}_{AB}) + \nu\Big(2\langle\bar{\beta}\rangle - 4\bar{\gamma}_{AB}\Big)\Big]$$

$$\delta_A \equiv \frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2} \qquad \epsilon_A \equiv \frac{(\beta'_A\alpha_B^3)^0}{(1+\alpha_A^0\alpha_B^0)^3} \qquad \zeta \equiv \frac{\beta_A^0\alpha_A^0\alpha_B^0\beta_B^0}{(1+\alpha_A^0\alpha_B^0)^3}$$

Recap :

• By means of a canonical transformation and a quadratic relation

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1+\frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

determined uniquely the effective Hamiltonian H_e.

• The whole 2PK dynamics has been reduced to the simple geodesic of an effective metric :

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \left[2\nu + \delta a_3^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^3 + \cdots$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \left[2(2-3\nu) + \delta b_2^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots$$

IV) ST – PARAMETRISED EOB DYNAMICS

• The inversion of the $H_e = f_{\rm EOB}(H)$ hence defines a unique, "resummed" EOB Hamiltonian :

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}$$

The dynamics deduced from $H_{\rm EOB}$ and the "real" Hamiltonians H are, by construction, equivalent up to 2PK order.

• $H_{\rm EOB}$ hence defines a resummed dynamics, that may capture some features of the strong field regime.

$$H_{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}\,, \quad \text{where} \quad H_e = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{r^2}\right)}$$

But H_{EOB} and H_e are conservative :

$$\Rightarrow \quad \boxed{\left(\frac{\partial H_{\mathrm{EOB}}}{\partial H_e}\right) = \frac{1}{\sqrt{1 + 2\nu(E-1)}}} \quad \text{since} \quad H_e = \mu E \quad \text{on-shell}$$

Hence the two-body eom, deduced from H_{EOB} are identical to the effective ones, deduced from H_e, to within a simple time rescaling

$$t \rightarrow t\sqrt{1 + 2\nu(E-1)}$$

.

In particular, the orbital frequency, deduced from $H_{\rm EOB}$, is

$$\Omega = \frac{\partial H_{\rm EOB}}{\partial H_e}\frac{\partial H_e}{\partial p_\phi} = \frac{j u^2 A}{G_{AB} M E\sqrt{1+2\nu(E-1)}} \qquad u = \frac{G_{AB} M}{r}$$

where, for circular orbits

$$j^2(u) = -\frac{A'}{(Au^2)'} \ , \quad E(u) = A\sqrt{\frac{2u}{(Au^2)'}}$$

and can be in particular evaluated at the level of the ISCO, $u_{\rm ISCO}$, such that :

$$\frac{A''}{A'} = \frac{(Au^2)''}{(Au^2)'}$$

→ **ST corrections to the ISCO location and frequency ?**
(typical near merger)

The ISCO location and frequency depend only on $A(u) = -g^e_{00}$

ST corrections to the strong-field regime

Last ingredient : the ST-corrected $A(u\,;\nu)$

$$u \equiv \frac{G_{AB} M}{r} \,, \quad \nu = \frac{m_A^0 m_B^0}{(m_A^0 + m_B^0)^2}$$

ST-corrected $A(u\,;\nu)$

$$A(u\,;\nu) = A^{\rm GR}_{\rm 2PN}(u\,;\nu) + 2\epsilon_{\rm 1PK} u^2 + (\epsilon^0_{\rm 2PK} + \nu \epsilon^\nu_{\rm 2PK}) u^3$$

where

$$\epsilon_{\rm 1PK} \equiv \langle \bar\beta \rangle - \bar\gamma_{AB}$$

$$\epsilon^0_{\rm 2PK} \equiv \frac{1}{12}\Big[-20\bar\gamma_{AB} - 35\bar\gamma_{AB}^2 - 24\langle\bar\beta\rangle(1-2\bar\gamma_{AB}) + 4\big(\langle\delta\rangle - \langle\epsilon\rangle\big)\Big]$$

$$\epsilon^\nu_{\rm 2PK} \equiv -3(\bar\beta_A + \bar\beta_B) + \frac{1}{3}\bar\gamma_{AB}(10+\bar\gamma_{AB}) + \frac{1}{3}(\epsilon_A + \epsilon_B) + \frac{2}{3}(\delta_A + \delta_B) - 2\zeta$$

ST-Corrections described by 3 parameters, $(\epsilon_{\rm 1PK}, \epsilon^0_{\rm 2PK}, \epsilon^\nu_{\rm 2PK})$

- **BUT** numerically driven by $(\alpha_A^0)^2$ (c.f. DEF, diagrammatic methods)
- When $(\alpha_A^0)^2 << 1$, $\boxed{\epsilon_{\rm 1PK} \sim \epsilon^0_{\rm 2PK} \sim \epsilon^\nu_{\rm 2PK}}$ and ST-corrections are perturbative

In this perturbative approach, **best available EOB-NR function** for GR :

$$A^{\rm GR}_{\rm 2PN}(u\,;\nu) \to \boxed{A^{\rm GR}_{\rm EOBNR}(u\,;\nu) = \mathcal{P}^1_5[A^{Taylor}_{\rm 5PN}]}$$

i.e. the $(1,5)$ Padé approximant of the truncated 5PN expansion :

$$A^{Taylor}_{\rm 5PN} = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + (a_5^c + a_5^{\ln} \ln u)u^5 + \nu(a_6^c + a_6^{\ln} \ln u)u^6$$

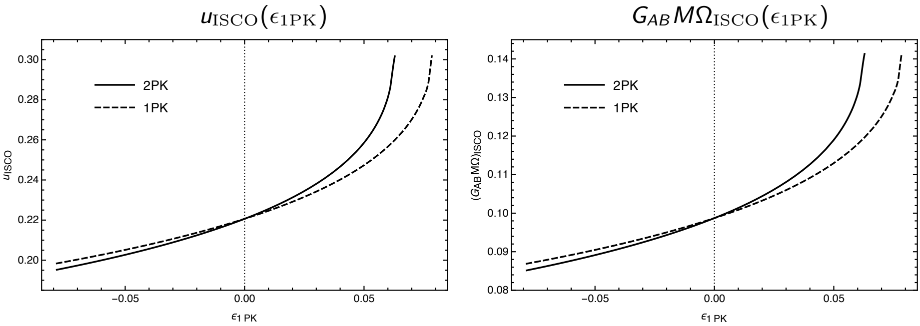
[Damour, Nagar, Reisswig, Pollney 2016]

- smoothly connected to Schwarzschild when $\nu \to 0$
- $a_6^c(\nu)$ is obtained by calibration with Numerical Relativity

ISCO Locations and frequency, equal-mass case ($\nu = 1/4$)

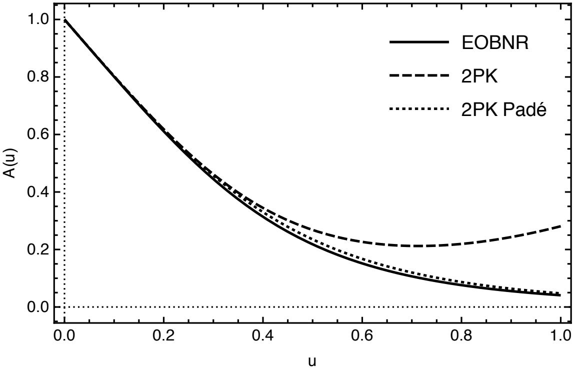
- 1PK corrections, $A = A^{\rm GR}_{\rm EOBNR}(u\,;\nu) + 2\epsilon_{\rm 1PK}\,u^2$
- 2PK corrections, $A = A^{\rm GR}_{\rm EOBNR}(u\,;\nu) + 2\epsilon_{\rm 1PK}\,u^2 + (\epsilon^0_{\rm 2PK} + \nu\epsilon^\nu_{\rm 2PK})u^3$

setting $\epsilon^0_{\rm 2PK} + \nu\epsilon^\nu_{\rm 2PK} \equiv \epsilon_{\rm 1PK}$



→ **dramatic increase** when $\epsilon_{\rm 1PK} \sim 10^{-1}$

$A(u) \quad (\nu = 1/4,\ \epsilon_{1\mathrm{PK}} = 0.08)$



- 2PK corrections, $A = A^{\mathrm{GR}}_{\mathrm{EOBNR}}(u\,;\nu) + 2\epsilon_{1\mathrm{PK}}\,u^2 + (\epsilon^0_{2\mathrm{PK}} + \nu\epsilon^\nu_{2\mathrm{PK}})u^3$
- 2PK Padeed corrections,

$$A = \mathcal{P}^1_5[A^{\mathrm{GR}}_{\mathrm{EOBNR}}(u\,;\nu) + 2\epsilon_{1\mathrm{PK}}\,u^2 + (\epsilon^0_{2\mathrm{PK}} + \nu\epsilon^\nu_{2\mathrm{PK}})u^3]$$

Tiny corrections to A, but significant corrections to the ISCO frequency

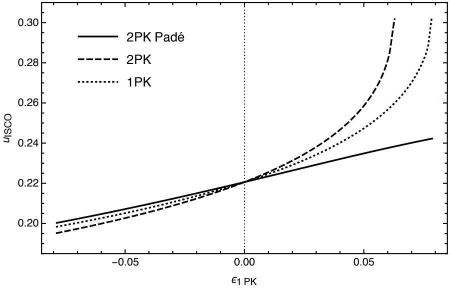
ST corrections to the strong-field regime

ISCO Locations and frequency, equal-mass case ($\nu = 1/4$)

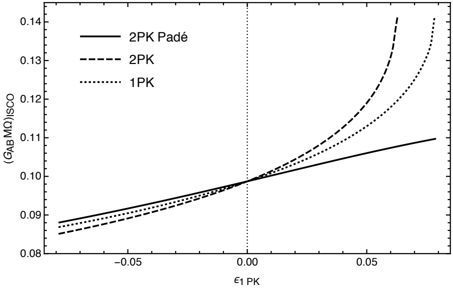
- 2PK Padeed corrections,

$$A = \mathcal{P}^1_5\left[A^{\rm GR}_{\rm EOBNR}(u\,;\nu) + 2\epsilon_{\rm 1PK}\,u^2 + (\epsilon^0_{\rm 2PK} + \nu\epsilon^\nu_{\rm 2PK})u^3\right]$$

$u_{\rm ISCO}(\epsilon_{\rm 1PK})$



$G_{AB}M\Omega_{\rm ISCO}(\epsilon_{\rm 1PK})$



$$\left.\frac{d(G_{AB}M\Omega)_{ISCO}}{d\epsilon_{\rm 1PK}}\right|_{\nu=1/4} \simeq 0.13$$

relative correction to GR significant ($\sim 10\%$) when $\epsilon_{\rm 1PK} \sim 10^{-2} - 10^{-1}$

Concluding remarks :

• Remarkably, the EOB approach is valid beyond the scope of General Relativity. In **Scalar-Tensor theories :**

$$A^{\rm 2PK}(u) \equiv \mathcal{P}^1_5[A^{Taylor}_{\rm 5PN} + 2\epsilon_{\rm 1PK} u^2 + (\epsilon^0_{\rm 2PK} + \nu\, \epsilon^\nu_{\rm 2PK})u^3]$$

• But also applicable for **any theory** whose coefficients $h_i^{\rm NPK}$ satisfy the 3 mapping conditions.

• The Scalar-Tensor example suggests a generic 2PK ansatz

$$\boxed{A^{\rm PEOB}(u) \equiv \mathcal{P}^1_5[A^{Taylor}_{\rm 5PN} + 2(\epsilon^0_{\rm 1PK} + \nu\, \epsilon^\nu_{\rm 1PK})u^2 + (\epsilon^0_{\rm 2PK} + \nu\, \epsilon^\nu_{\rm 2PK})u^3]}$$

where $\epsilon^0_{\rm 1PK}$, $\epsilon^\nu_{\rm 1PK}$, $\epsilon^0_{\rm 2PK}$, and $\epsilon^\nu_{\rm 2PK}$ are theory-agnostic Parametrized EOB (PEOB) coefficients.

Conclusion

• The EOB approach has been extended to neutron stars including **tidal effects** (TEOB, [Damour, Nagar 2010]) through 5PN $\mathcal{O}(u^6)$ corrections to $A(u)$. To be compared with ST-EOB $\mathcal{O}(u^2)$.

• Binary pulsar experiments have put **stringent constraints on ST theories** (no dipolar radiation)

$$\boxed{(\alpha_A^0)^2 < 4 \times 10^{-6}}$$

For **any** body A, regardless of its EOS or self-gravity.

• The ISCO ST-correction (significant for $(\alpha_A^0)^2 \gtrsim 10^{-2}$) seems unlikely to improve binary pulsar constraints.

However :

• However, stars subject to dynamical scalarization can develop non perturbative $(\alpha_A^0)^2$ near merger [Barausse, Palenzuela, Ponce, Lehner 2013]. EOB well-suited to investigate this regime !

• The interferometers LIGO-Virgo or even LISA are designed to detect highly redshifted sources. Cosmological history of ST theories ?

Black holes :

• Are known in these ST theories to carry no scalar hair : $m_A(\varphi) = cst$ i.e. no deviation to GR.

• Induce scalar hair by means of $V(\varphi)$ or (vector) gauge field ? (at least 1PK by Lorentz invariance).

• Conditions of no hair theorems may not apply anymore in the strong field, highly dynamical regime of a merger, which is **precisely** explored by the EOB approach !