Gravitational and statistical physics to model the propagation of light in a (more) realistic universe

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- All the information about our universe is (so far) obtained (mostly) through the analysis of electromagnetic radiation.

- Propagation of light is thus central in the construction of a cosmological model since it links the predictions to actual observations.

- Propagation of light enters in all distance measurements and all position measurements.

- It determines the lightcone structure and thus the nature of particle/event horizons.
Equations for a beam

\[ \nabla_\mu F^{\mu \nu} = 0 \quad \Box A^\mu = R_\nu^\mu A^\nu \]

- Eikonal approximation
  \[ k_\mu k^\mu = 0 \quad k^\mu \nabla_\mu k^\nu = 0 \]

- Thin beam approximation
  \[ k^\mu \xi_\nu = 0 \quad \frac{d^2 \xi^\mu}{d\lambda^2} = R^\mu_{\nu \alpha \beta} k^\nu k^\alpha \xi^\beta \]
Equations for a beam

\[ \frac{d^2 \xi^a}{d\lambda^2} = \mathcal{R}_{ab} \xi^b \]

with

\[ \mathcal{R}_{ab} = R_{\mu \nu \alpha \beta} k^\nu k^\alpha n^\mu_a n^\beta_b \]

The linearity of this equation implies that

\[ \xi^a(\lambda) = \mathcal{D}^a_b(\lambda) \left. \frac{d\xi^b}{d\lambda} \right|_{\lambda=0} \]

<table>
<thead>
<tr>
<th>Sachs equation</th>
<th>Initial condition</th>
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<tbody>
<tr>
<td>[ \frac{d^2}{dv^2} \mathcal{D}^a_b = \mathcal{R}^a_c \mathcal{D}^c_b ]</td>
<td>[ \mathcal{D}^a_b(0) = 0, \quad \frac{d\mathcal{D}^a_b}{dv}(0) = \delta^a_b ]</td>
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\[ (\mathcal{D}^A_B) = \frac{D_A}{\text{distance}} \cdot (\cos \psi \quad \sin \psi \quad -\sin \psi \quad \cos \psi) \cdot \exp \left( \begin{pmatrix} -\Gamma_1 & \Gamma_2 \\ \Gamma_2 & -\Gamma_1 \end{pmatrix} \right) \cdot \mathcal{D}_{ab} \equiv \kappa I_{ab} + V \varepsilon_{ab} + \gamma_{ab} \]
Effects of the small scale structures
Thin beam in cosmology

- Lensing can influence in particular the Hubble diagram obtained from supernovae

- In the standard lore, one takes this effect perturbatively into account.
Is this the correct method?

Supernovae subtend a very thin beam (typical 1μas).

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How homogeneous is homogeneous?

Small scales (e.g. ~AU) cannot be reached by numerical simulations. Find some approximate exact solutions

Fleury, Dupuy, JPU, arXiv:1302.5308
Experimenting with a Swiss cheese model

[Fleury, Uzan, Dupuy, 2013-14]
How homogeneous is homogeneous?

Small scales (e.g. ~AU) cannot be reached by numerical simulations. Find some approximate exact solutions.

Fleury, Dupuy, JPU, arXiv:1302.5308
Extending CP: how homogeneous is homogeneous?

\[
\frac{D_L - D_L^{(FL)}}{D_L^{(FL)}} \times 100
\]

\[z = 0.1\]

\[f = 0.9\]

\[f = 0.7\]

\[f = 0.26\]
Extending CP: how homogeneous is homogeneous?

Fitting with a mock Hubble diagram with a FL luminosity distance

Fitting with SNLS 3 data with an approximate luminosity distance

Fleury, Dupuy, JPU, arXiv:1304.7791
Extending CP: how homogeneous is homogeneous?

Fleury, Dupuy, JPU, arXiv:1304.7791
Stochastic lensing

[Fleury, Larena, Uzan, 2015]
Toward an analytical understanding of the dispersion

- The average angular distance does not follow the Dyer-Roeder equation

- To describe the dispersion we assume that
  - *Small-scal lensing* ~ diffusion process
  - *Ricci & Weyl lensing* ~ white noise
The Sachs-Langevin equation

\[ \frac{d^2 D}{dv^2} = (\langle R \rangle + \delta R) D \]

- \( \langle R \rangle \) (deterministic) encodes the large-scale structure;
- \( \delta R \) (stochastic) models small-scale fluctuations.

**Hypotheses**: statistical isotropy and white noises.

\[ \langle W_A \rangle = 0, \]
\[ \langle \delta R(v) W_A(w) \rangle = 0, \]
\[ \langle \delta R(v) \delta R(w) \rangle = C_R(v) \delta(v - w) \]
\[ \langle W_A(v) W_B(w) \rangle = C_W(v) \delta_{AB} \delta(v - w) \]

Covariance amplitude of \( X \) such that \( C_X \sim (\delta X)^2 \Delta v_{coh} \).
Toward an analytical understanding of the dispersion

\[
\frac{d^2 D_A}{d\lambda^2} = (\langle R \rangle + \delta R - |\sigma|^2) \, D_A
\]

\[
\frac{dD_A \sigma}{d\lambda} = \mathcal{W} D_A^2
\]

- \(D_A\) and \(\sigma\) are now stochastic processes
- Everything is encoded in the covariances of \(\delta R\) and \(W\).
- It allows to derive a Fokker-Planck equation
The lensing Fokker-Planck equation

[Fleury, Larena, Uzan 2015]

The probability density function $p(D, \dot{D}; \nu)$ satisfies

$$\frac{\partial p}{\partial \nu} = -\dot{D}_{AB} \frac{\partial p}{\partial D_{AB}} - \langle R \rangle D_{AB} \frac{\partial p}{\partial \dot{D}_{AB}}$$

$$+ \frac{1}{2} \left( C_\mathcal{R} \delta_{AE} \delta_{CF} + C_\mathcal{W} \delta_{AC} \delta_{EF} - C_\mathcal{W} \varepsilon_{AC} \varepsilon_{EF} \right) D_{EB} D_{FD} \frac{\partial^2 p}{\partial \dot{D}_{AB} \partial \dot{D}_{CD}},$$

with a drift term and a diffusion term.

- It generates evolutions equations for the moments of $p(D, \dot{D}; \nu)$.
- Order-$n$ moments form a closed system (no hierarchy).
- Everything is contained in the functions $\langle R \rangle (\nu)$, $C_\mathcal{R}(\nu)$, $C_\mathcal{W}(\nu)$. 
Comparison to Swiss-Cheese model

- Take a Swiss-cheese simulation
- Compare to the analytical computation
  (in that case the correlation functions of $\delta R$ and $W$ can be computed)

- How can one relate this to the distribution of the voids?
Application to Swiss-cheese models

- We use the Swiss-cheese model as a test of the method.
- $\langle \mathcal{R}(v) \rangle$, $C_{\mathcal{R}}(v)$, $C_{\mathcal{W}}(v)$, can be calculated analytically.
- Results on $\delta^{(1)}_{D_A}$ and $\sigma_{D_A}$:
The problem of non-Gaussianity

- The Fokker-Planck approach assumes **Gaussian** noises.
- It is motivated by the central limit theorem.
- Here the convergence towards the central limit is too slow ($\sim 10^3$ deflections, whereas Brownian motion has $10^{20}$ collisions/second).

First indication: reducing the non-Gaussianity of the Weyl noise yields
Ricci-Weyl

[Fleury, Larena, Uzan, 2017]
Ricci focuses due to diffuse matter inside the beam, as
\[
\mathcal{R} = -4\pi G T_{\mu\nu} k^\mu k^\nu = -4\pi G \omega^2 (\rho + p).
\]
Weyl distorts and focuses mostly due to matter outside the beam.
Ricci focuses due to diffuse matter inside the beam, as

$$\mathcal{R} = -4\pi GT_{\mu\nu}k^\mu k^\nu = -4\pi G\omega^2(\rho + p).$$

Weyl distorts and focuses mostly due to matter outside the beam.

Lensing smoothes the matter distribution on the beam. External sources does not affect the area.
Strong lensing in the weak regime

- Describe the beam with the strong lensing regime

- Assume that lenses are larger than their Einstein radius
  - only 1 image
  - \( \epsilon_i |\theta - \theta_i| \ll 1 \)

\[
\epsilon_i^2 = \frac{4GM_i}{c^2} \frac{D_{Li}S}{D_{OS_i}D_{OL}}
\]
Strong lensing in the weak regime

\[ \beta = \theta - \sum_{i=1}^{N} \varepsilon_i^2 \frac{\theta - \theta_i}{|\theta - \theta_i|^2} \]

\[ s = z - \sum_{i=1}^{N} \frac{\varepsilon_i^2}{z^* - w_i^*} \]

The area of the image is given by

\[ \Omega = \frac{1}{2i} \int_{\mathcal{I}} z^* \, dz. \]
Strong lensing in the weak regime

\[ \Omega = \frac{1}{2i} \int_I z^* \mathrm{d}z. \]

\[ \Omega = \frac{1}{2i} \int_S s^* \mathrm{d}s + \frac{1}{2i} \sum_{i=1}^N \varepsilon_i^2 \int_I \frac{\mathrm{d}z}{z - w_i} \]

\[ - \frac{1}{2i} \sum_{i=1}^N \varepsilon_i^2 \left[ \int_I \frac{z\mathrm{d}z}{(z - w_i)^2} \right]^* + \mathcal{O}(\varepsilon^4) \]

\[ \Omega = \Omega_S + \sum_{i \in \mathcal{B}} \frac{4\pi G m_i D_{iS}}{D_{OS} D_{Oi}} + \mathcal{O}(\varepsilon^4) \]
Strong lensing in the weak regime

\[ \Omega = \Omega_S + \sum_{i \in B} \frac{4\pi G m_i D_{iS}}{D_{OS} D_{Oi}} + \mathcal{O}(\varepsilon^4) \]

For an infinitesimal beam \( \kappa \equiv \Omega / \Omega_S - 1 \)

\[ \kappa = \frac{1}{2} \int_0^\lambda \frac{\lambda' (\lambda - \lambda')}{\lambda} (R_{\mu\nu} k^\mu k^\nu) \, d\lambda' . \]

\[ \rho(r) = \sum_{i=1}^N \frac{m_i}{A_i} \delta(\lambda - D_i) , \quad A_i = D_i^2 \Omega \]

For convergence is given only by the masses inside the beam and does not depend on the way the matter is distributed.
Strong lensing in the weak regime

\[ Q_{ab} = \int_{0}^{2\pi} \left[ \theta_a(\varphi)\theta_b(\varphi) - \beta^2 \right] d\varphi \]
Conclusions

- Lensing has become a standard tool in cosmology and several observational projects are ongoing.

- Beyond the standard analysis, it allows one:
  - To test general relativity [JPU & Bernardeau, 2001]
  - To test the isotropy of expansion at percent level [Pitrou, Pereira, JPU, 2013]

- The effect of some scale structures on thin beam drives many questions
  - New stochastic lensing formalism [Fleury, Larena, JPU, 2015]
  - How to relate the dispersion of the Hubble diagram to the distribution of voids.