BMS Vacua and Black Holes

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Recent Developments in General Relativity
In memory of Joseph Katz (1930-2016)
Jerusalem, May 21, 2017

Joseph Katz' resolution of Komar's discrepancy

Class. Quantum Grav. **2** (1985) 423–425. Printed in Great Britain

COMMENT

A note on Komar's anomalous factor

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Received 26 October 1984

Abstract. Discrepancies between Komar's integrals for energy and angular momentum and those given by Einstein are simply explained. In particular the origin of the radiative anomaly is found.

It is well known that Komar's (1959) conserved quantities

$$K \equiv \int_\Sigma \hat{K}^\mu \, d\Sigma_\mu = \int_\Sigma \frac{2}{\kappa} D_\nu D^{[\mu} \hat{\xi}^{\nu]} \, d\Sigma_\nu = \int_S \frac{2}{\kappa} D^{[\mu} \hat{\xi}^{\nu]} \, dS_{\mu\nu}, \qquad \kappa = 8\pi G \tag{1}$$

The puzzle

It is well known that Komar's (1959) conserved quantities

$$K \equiv \int_\Sigma \hat{K}^\mu \, \mathrm{d}\Sigma_\mu \equiv \int_\Sigma \frac{2}{\kappa} D_\nu D^{[\mu} \hat{\xi}^{\nu]} \, \mathrm{d}\Sigma_\nu = \int_S \frac{2}{\kappa} D^{[\mu} \hat{\xi}^{\nu]} \, \mathrm{d}S_{\mu\nu}, \qquad \kappa = 8\pi G \tag{1}$$

(where Σ is a timelike hypersurface, say $x^0 =$ constant, with boundary S, a sphere at infinity and ^ means multiplication by $\sqrt{-g}$) has the following properties.

(i) Regarding Schwarzschild and Kerr black holes, if $\xi^\nu = (1, 0, 0, 0)$ in (t, r, θ, ϕ) coordinates, then K is equal to the mass M.

(ii) For Kerr black holes and $\xi^\nu = (0, 0, 0, 1)$ in Boyer–Linquist coordinates, K is *twice* the angular momentum J. This is the anomalous factor 2.

(iii) For the Bondi *et al* (1962) radiating asymptotic solution, if ξ^ν is asymptotically $(1, 0, 0, 0)$ in radiative coordinates $(u = t - r,\ r,\ \theta,\ \phi)$, K is *not* Bondi's mass $M(u, \theta)$ but rather $M + \frac{1}{2} c\dot{c}$ in Bondi's notation. Winicour and Tamburino (1965) have found a correction to Komar's integral which gets rid of the $c\dot{c}$ term.

On the other hand, it is also well known (Møller 1972) that Einstein's pseudo-tensor of energy momentum gives, in appropriate asymptotic coordinates, the expected results, namely M in case (i), J in case (ii) and $M(u, \theta)$ in case (iii).

The method

On the other hand, it is also well known (Møller 1972) that Einstein's pseudo-tensor of energy momentum gives, in appropriate asymptotic coordinates, the expected results, namely M in case (i), J in case (ii) and $M(u, \theta)$ in case (iii).

The burden of having to choose appropriate coordinates disappears if we use the formal tricks of an artificially introduced flat background (Cornish 1964); then Einstein's conserved pseudo-tensor becomes a real tensor. Therefore, in this 'bimetric' formalism, it should be possible to find a relation between Komar's covariant conservation laws and Einstein's covariant conserved energy tensor. The relation is indeed simple and is given below.

If besides $g_{\mu\nu}$, we postulate the existence of some flat background $\bar{g}_{\mu\nu}$ whose points, with the same coordinates x^λ, are in one-to-one correspondence with the points of the curved space (see Rosen 1963) then we shall meet three types of derivatives: partial derivatives ∂_λ, covariant derivatives D_λ and 'covariant-flat' derivatives $\bar{\mathrm{D}}_\lambda$. Furthermore, the following identities for Christofel symbols $\Gamma^\lambda_{\mu\nu}$ and Ricci's tensor $R_{\mu\nu}$ hold

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\rho}(\bar{\mathrm{D}}_{(\mu}g_{\nu)\rho} - \tfrac{1}{2}\bar{\mathrm{D}}_\rho g_{\mu\nu}) + \bar{\Gamma}^\lambda_{\mu\nu} \equiv \Delta^\lambda_{\mu\nu} + \bar{\Gamma}^\lambda_{\mu\nu} \qquad (2)$$

and

$$R_{\mu\nu} = \bar{\mathrm{D}}_\lambda \Delta^\lambda_{\mu\nu} - \bar{\mathrm{D}}_\mu \Delta^\lambda_{\nu\lambda} + \Delta^\rho_{\mu\nu}\Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma}\Delta^\sigma_{\rho\nu} + \bar{R}_{\mu\nu}. \qquad (3)$$

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The resolution

Since the background is flat, we have of course

$$\bar{R}_{\mu\nu} = 0. \tag{4}$$

By contracting (3) with $\hat{g}^{\mu\nu}$, we shall obtain the bimetric version of a very familiar equality

$$\hat{\mathscr{L}} \equiv \hat{R} + \partial_\lambda \hat{k}^\lambda = -\hat{g}^{\mu\nu}(\Delta^\rho_{\mu\nu}\Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma}\Delta^\sigma_{\rho\nu}) \tag{5}$$

in which $\hat{k}^\lambda$ is a vector density

$$\hat{k}^\lambda \equiv (\sqrt{-g})^{-1}\bar{\mathrm{D}}_\sigma \hat{\hat{g}}^{\lambda\sigma} \tag{6}$$

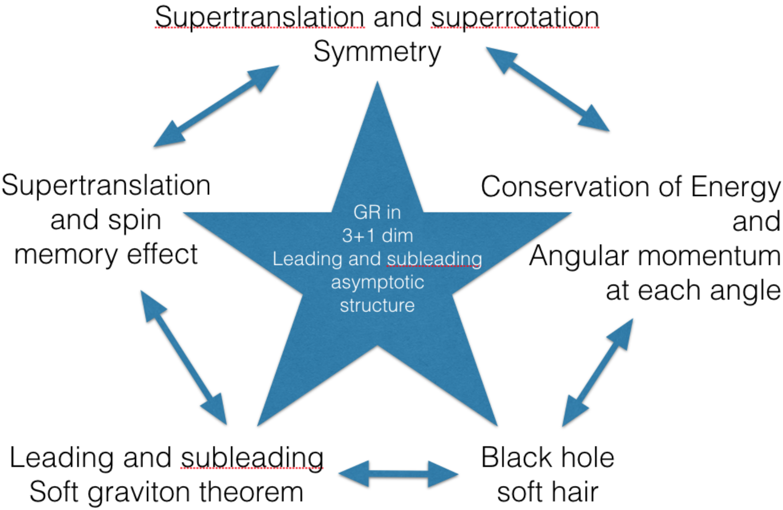
$\hat{\mathscr{L}}$ is a Lagrangian density for Einstein's equations.

By adding both contributions we thus find that

$$\hat{J}^{[\mu\nu]} = \hat{F}^{[\mu\nu]}_\lambda \xi^\lambda + \kappa^{-1}\hat{g}^{\sigma[\mu}\delta^{\nu]}_\lambda \bar{\mathrm{D}}_\sigma \xi^\lambda = \kappa^{-1}(\mathrm{D}^{[\mu}\hat{\xi}^{\nu]} + \xi^{[\mu}\hat{k}^{\nu]}). \tag{15}$$

Pioneer formula which will be derived in covariant phase space formalism by [Lee, Wald, 1991] [Iyer, Wald, 1994].

Summary : BMS and IR structure of gravity



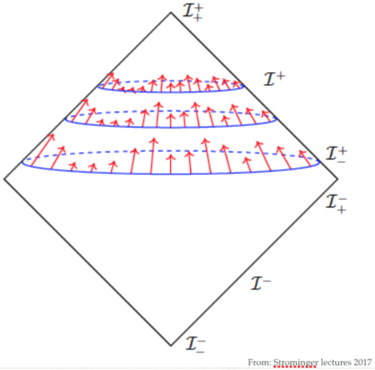
[Strominger et al. 2013-2017]

Supertranslations

$$u \to u + T(\theta, \phi)$$

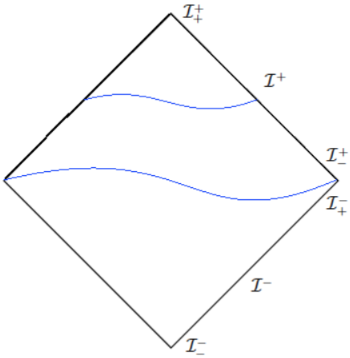
- The 4 lowest spherical harmonics of T correspond to time and spatial translations.
- The highest harmonics are the supertranslations

[Bondi, van der Burg, Metzner ; Sachs, 1962]



Superrotations

$$z = e^{i\phi} \cot \frac{\theta}{2} \qquad \begin{array}{l} z \to F(z) \\ \bar{z} \to F(\bar{z}) \end{array}$$



- The 3 complex $1, z, z^2$ correspond to Lorentz transformations

$$SO(3,1)^{\uparrow}_{+} \simeq SL(2,\mathbb{C})$$

- The other (singular) meromorphic transformations are the superrotations.

[Penrose, Nutku, 1972] [de Boer, Solodukhin, 2003] [Barnich, Troessaert, 2010]

The final state of collapse in Einstein gravity

The final metric after gravitational collapse, if analytic, is diffeomorphic to the Kerr metric (M,J). [No hair theorems]

[Carter, Hawking, Robinson, 1971-1975] [Chrusciel, Costa, 2008] [Alexakis, Ionescu, Klainerman, 2009]

But diffeomorphisms can be non-trivial : associated with non-vanishing canonical charge : a superrotation charge, which could be evaluated around the black hole horizon.

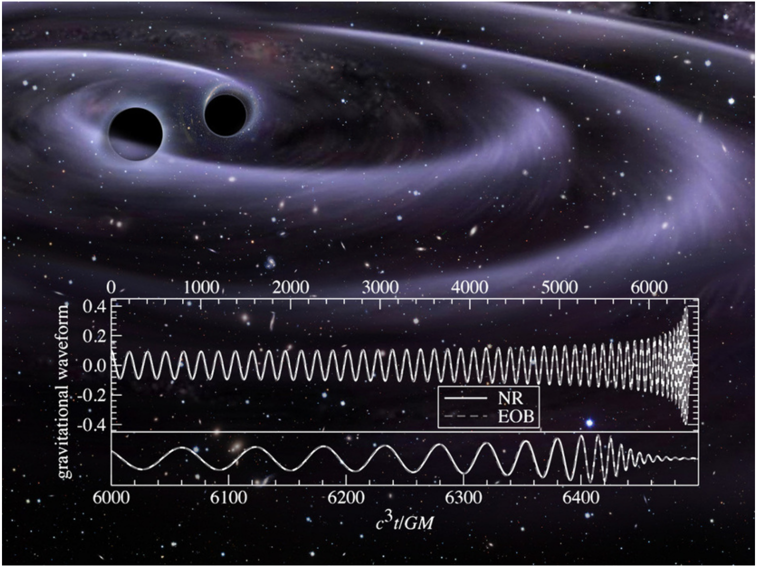
[Bondi, van der Burg, Metzner ; Sachs, 1962] [Barnich, Troessaert, 2010] [Compère, Long, 2016]

The final state of collapse depends upon (M,J) and the *supertranslation memory field* $C(\theta,\phi)$ which encodes distinct BMS Poincaré vacua.

[Compère, Long, 2016] [Hawking, Perry, Strominger, 2016]

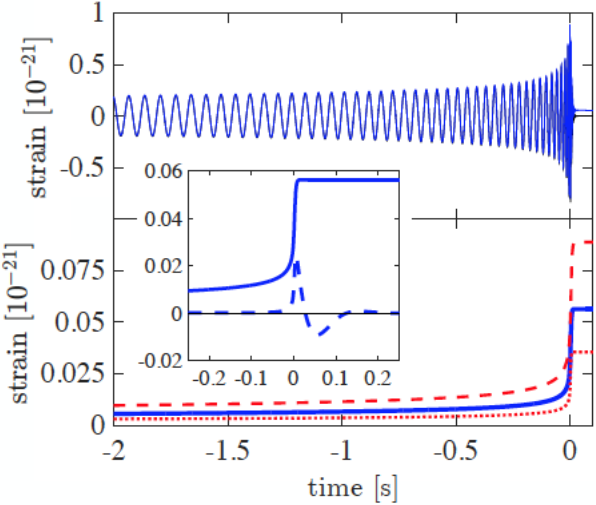
Signature in Hawking's radiation ?

GW150914



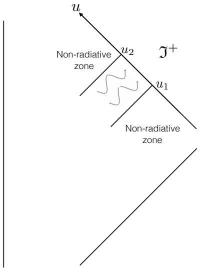
[Image credit : UMD/AEI/Milde Marketing/ESO/NASA]

Gravitational Memory of GW150914



[Reproduced from Lasky at al., 2016]

The memory effect is understood around $\mathfrak{I}^+$



After the passage of either gravitational waves or null matter between two detectors placed around $\mathfrak{I}^+$, the detectors acquire a finite relative spacetime displacement.

This is a 2.5PN General Relativity effect. [Damour, Blanchet, 1988]

It is refered to as the linear memory effect when caused by a change of mass aspect or passage of null matter [Zeldovich, Polnarev, 1974] and the non-linear memory or Christodoulou effect when caused by gravitational waves [Christodoulou, 1991].

The memory effect

Einstein's equations integrated over a finite retarded time interval of $\mathfrak{I}^+$:

$$-\frac{1}{4}\nabla^2(\nabla^2+2)(C|_{u_2}-C|_{u_1}) = m|_{u_2} - m|_{u_1} + \int_{u_2}^{u_1} du T_{uu},$$

$$T_{uu} \equiv \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T_{uu}^{matter}].$$

This is the angle-dependent energy conservation law. [Geroch, Winicour, 1980] [Frauendiener, 1992] [Christodoulou, Klainerman, 1993] [Strominger, Zhiboedov, 2014]

The shift $C|_{u_2} - C|_{u_1}$ can be constructed from the radiation flux history. It allows to compute the shift of the geodesic deviation vector s^A, $A = \theta, \phi$ as

$$s^A|_{u_2} - s^A|_{u_1} \sim \frac{1}{r}\partial^A\partial_B(C|_{u_2} - C|_{u_1})s^B.$$

The supertranslation memory field

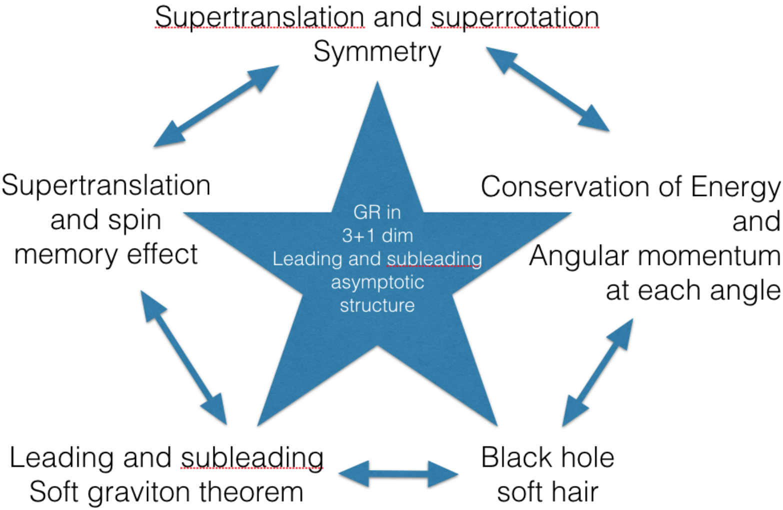
The phase space of Einstein gravity contains in particular one canonical variable best identified at null infinity in stationary regions called the supertranslation memory field

$$C(\theta, \phi)$$

Under a supertranslation, it changes as

$$\delta_T C(\theta, \phi) = T(\theta, \phi).$$

Summary : BMS and IR structure of gravity



[Strominger et al. 2013-2017]

Bulk extension of asymptotic symmetries

For describing black hole soft hair, asymptotic symmetries are not enough. Now, asymptotic symmetries can be continued into the bulk of spacetime in two ways :

- The symmetry algebra $[\xi^a, \xi^b]$ can be extended in the bulk of spacetime

  [Barnich, Troessaert, 2009]

- In stationary configurations, the conserved charges $\mathcal{Q}_\xi$ are also conserved in the bulk of spacetime

  [Compère, Donnay, Lambert, Schulgin, 2014] [Compère, Hajian, Seraj, Sheikh-Jabbari, 2015]

Bulk extension of asymptotic symmetries

The generalized Noether theorem for diffeomorphism invariant theories [Iyer, Wald , 1994] [Barnich, Brandt, Henneaux, 1995] [Barnich, Brandt, 2001]

$$\boldsymbol{\omega}[\mathcal{L}_\xi g_{\mu\nu}, \delta g_{\mu\nu}; g_{\mu\nu}] \approx \boldsymbol{d}\boldsymbol{k}_\xi[\delta g_{\mu\nu}; g_{\mu\nu}]$$

and Stokes' theorem then leads to conserved charges

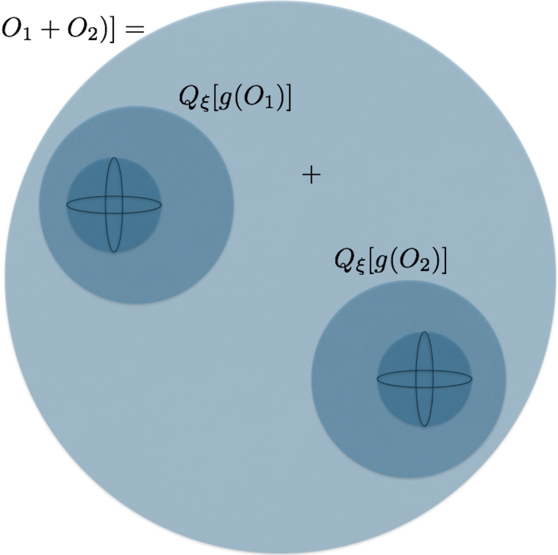
$$Q_\xi[g] = \int_{\bar{g}}^{g} \int_S \boldsymbol{k}_\xi[\delta g_{\mu\nu}; g_{\mu\nu}]$$

either for

- Killing vectors ($\mathcal{L}_\xi g_{\mu\nu} = 0$)
- Asymptotic Killing vectors ($\mathcal{L}_\xi g_{\mu\nu} \to 0$ at the boundary)
- *Symplectic symmetries* $\boldsymbol{\omega}[\mathcal{L}_\xi g_{\mu\nu}, \delta g_{\mu\nu}; g_{\mu\nu}] \approx 0$

Additivity of symplectic charges

$Q_\xi[g(O_1 + O_2)] =$



[Barnich, 2003] [Compère, Hajian, Seraj, Sheikh-Jabbari, 2015]

Building BMS vacua and black holes

[G.Compère, J. Long, 2016]

Algorithm :

- Start with Minkowski/Schwarzschild spacetime.
- Write the change of coordinates $x^\mu \to x'^\mu$ which exponentiates the infinitesimal change of coordinates $x'^\mu = x^\mu + \xi^\mu + \dots$ where ξ^μ is a generic *s*upertranslation vector field at $\mathfrak{I}^+$
- Solve for the finite change of coordinates at each order in the asymptotic radial expansion such that $g_{\mu\nu} = \frac{\partial x^\beta}{\partial x'^\mu} \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\nu}$ fits in Newman-Unti gauge.
- Resum the infinite radial expansion.
- Rewrite the final metric in closed form.

The result is the BMS orbit of Minkowski/Schwarzschild spacetime. It is the representation of the BMS group on the bulk metric.

The Poincaré vacua of Einstein gravity

The vacuum metric with supertranslation field only is

$$ds^2 = -dt^2 + dx_s^2 + dy_s^2 + dz_s^2 = -du^2 - 2dudr + g_{AB}d\theta^A d\theta^B,$$

where $\theta^A = \theta, \phi$ and

$$\begin{aligned} g_{AB} &= (r-C)^2\gamma_{AB} - 2(r-C)D_A D_B C + D_A D_E C D_B D^E C, \\ &= (r\gamma_{AC} - D_A D_C C - \gamma_{AC} C)\gamma^{CD}(r\gamma_{DB} - D_D D_B C - \gamma_{DB} C) \end{aligned}$$

Under a supertranslation,

$$\delta_T C(\theta, \phi) = T(\theta, \phi).$$

It admits 10 Killing vectors. We checked that the 10 Poincaré charges are zero $\Rightarrow$ Poincaré vacua.

All supertranslation charges are zero.

The Poincaré vacua of Einstein gravity

The vacua are non-trivial in the sense that they admit canonical charges : superrotation charges

$$\mathcal{Q}_R = -\frac{1}{4G}\int_S d^2\Omega R^A \left(\frac{1}{8}D_A(C_{EF}C^{EF}) + \frac{1}{2}C_{AB}D_E C^{EB}\right)$$

where $C_{AB} = -2D_A D_B C + \gamma_{AB} D^2 C$. [Barnich, Troessaert, 2011]

Even thought the superrotation transformation $\delta z = R^z(z)$, $\delta\bar{z} = R^{\bar{z}}(\bar{z})$ admit poles, the superrotation charges are finite.

The superrotation charges are conserved at finite radius r. Superrotations are symplectic symmetries.

There is therefore an obstruction in the bulk at shrinking the surface of integration $\Rightarrow$ Bulk source for the superrotation charge.

Mapping Symmetries from $\mathfrak{I}^+$ to $\mathfrak{I}^-$

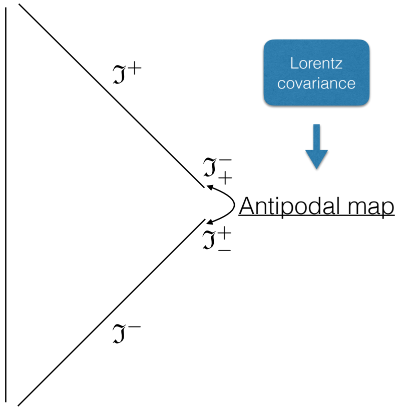
Requiring BMS invariance of the S-matrix and consistency with Weinberg soft graviton theorem requires a map between BMS supertranslations at $\mathfrak{I}^-$ and $\mathfrak{I}^+$

$$BMS_+ \times BMS_- \to BMS_{diagonal}$$

[Strominger, 2013] [He, Lysov, Mitra, Strominger, 2014]

Similarly to Poincaré symmetry, there is only one BMS symmetry asymptotically.

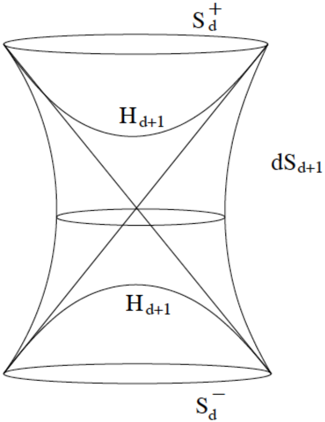
Boundary condition at spatial infinity



Observed for the mass aspect [Herberthson, Ludvigsen, 1992]
From Lorentz covariance and consistency of scattering [Strominger, 2013]
From boundary conditions [Troessaert, April 2017]

Asymptotic structure at spatial infinity

$d + 2$ dimensions :



[Ashtekar, Hansen, 1978][Beig, Schmidt, 1982] [de Boer, Solodukhin, 2003]

Note : At the boundary of AdS_3 (and H_3 and dS_3) there are two copies of the Virasoro algebra. [Brown, Henneaux, 1986]
This leads to $4d$ superrotations. [Barnich, Troessaert, 2010]

$3d$ Einstein gravity as a toy model for BMS Vacua

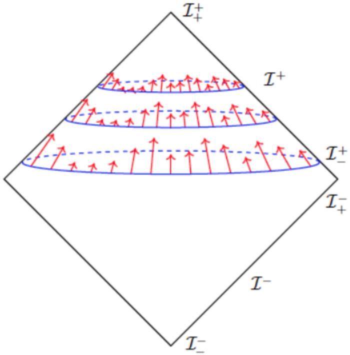
- No gravitational wave
- No black hole. [Ida, 2000]
- Yet : Non-trivial diffeomorphisms
- Yet : Minkowski vacuum and BMS Vacua
- Yet : Conical defects $0 < \Delta \leqslant 1$. The mass is $M = -\frac{\Delta^2}{8G}$.
- Yet : Other solutions (cosmological solutions, conical excesses, ...)

3*d* Supertranslations

$$u \to u + T(\phi)$$

- The 3 lowest Fourier modes of T correspond to time and spatial translations.
- The higher Fourier modes are the supertranslations

[Ashtekar, Bicak, Schmidt 1997 ; Barnich, Compère, 2006]



From: Strominger lectures 2017

$3d$ Superrotations

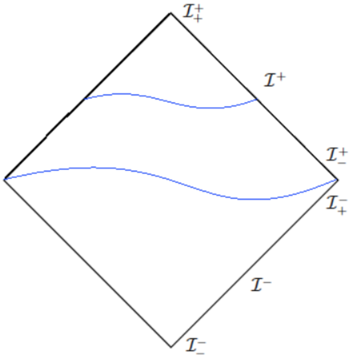
$$\phi \to \phi + R(\phi)$$

- The 3 lowest Fourier modes correspond to Lorentz transformations

$$SO(2,1)_+^\uparrow \simeq SL(2,\mathbb{R})$$

- The higher (regular) Fourier modes correspond to superrotations.

[Ashtekar, Bicak, Schmidt 1997 ; Barnich, Compère, 2006]



Linking $\mathcal{I}^+$ and $\mathcal{I}^-$: the 3*d* case

Main results :

- Boundary conditions exist which admit BMS_3 (supertranslations and superrotations) as symmetry group
- The entire metric is a representation of BMS_3. Symmetries are asymptotic at both null infinities, at spatial infinity and in the bulk of spacetime (symplectic)
- Fields defined at $\mathcal{I}^+$ are related to the ones at $\mathcal{I}^-$ by the antipodal map $\phi \to \phi + \pi$ in the absence of a global conical defect.

[Compère, Fiorucci, May 17, 2017]

Solution space of $3d$ Einstein gravity

The general solution in the hyperbolic foliation can be written in closed form as

$$ds^2 = d\rho^2 + (\rho^2 h^{(0)}_{ab} + \rho h^{(1)}_{ab} + h^{(2)}_{ab}) dx^a dx^b.$$

where

$$h^{(1)}_{ab} = T_{ab} - h^{(0)}_{ab} h^{cd}_{(0)} T_{cd},$$
$$h^{(2)}_{ab} = \frac{1}{4} h^{(1)}_{ac} h^{cd}_{(0)} h^{(1)}_{db},$$

in terms of 2 holographic ingredients : $h^{(0)}_{ab}$, T^{ab} obeying

$$R_{(0)} = 2, \qquad \mathcal{D}_a T^{ab} = 0.$$

[de Boer, Solodukhin, 2003]

Residual diffeomorphisms

Boundary diffeomorphisms

$$\xi^a_{(0)}(x^b)\partial_a$$

Spi-supertranslations

$$\omega(x^b)\partial_\rho + O(\rho^{-1})\partial_a$$

Boundary causality : 2d de Sitter spacetime

- Global dS_2

$$ds^2_{(0)} = -d\tau^2 + \cosh^2\tau d\phi^2 = \frac{-dT^2 + d\phi^2}{\cos^2 T}$$

where

$$\phi \sim \phi + 2\pi, \qquad -\frac{\pi}{2} < T < \frac{\pi}{2}$$

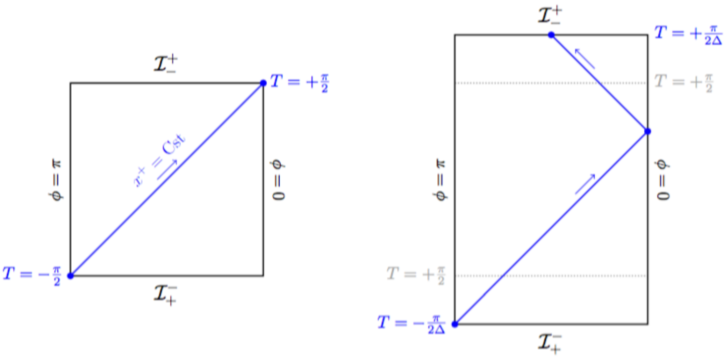
- Global dS_2 with conical defect $0 < \Delta \leqslant 1$

$$ds^2_{(0)} = -d\tau^2 + \Delta^2\cosh^2\tau d\phi^2 = \Delta^2\frac{-dT^2 + d\phi^2}{\cos^2(\Delta T)}$$

where

$$\phi \sim \phi + 2\pi, \qquad -\frac{\pi}{2\Delta} < T < \frac{\pi}{2\Delta}.$$

Null geodesics obey the antipodal map



(a) *Without conical defect* ($\Delta = 1$). (b) *With conical defect* ($0 < \Delta < 1$).

Figure 1: Penrose diagram illustrating null geodesic motion on the dS_2 boundary in the absence or presence of a global conical defect.

Boundary conditions leading to BMS_3

Motivated by the properties of null geodesics, we define null boundary coordinates as

- $h^{(0)}_{++} = h^{(0)}_{--} = 0$.

Existence of a variational principle then holds for

- $T^a_a = 0$.

We further require

- Asymptotic flatness at $\mathcal{I}^+$ and $\mathcal{I}^-$ in the sense of [Barnich, Compère, 2006]

Double copy of BMS$_3$

Superrotations which preserve null coordinates are

$$R^+(x^+)\partial_+ + R^-(x^-)\partial_-$$

Spi-supertranslations which preserve $T^a_a = 0$ obey $(\mathcal{D}_a\mathcal{D}^a + 2)\omega = 0$. The explicit solution depends upon

$$T^+(x^+), \qquad T^-(x^-).$$

It defines left and right supertranslations.

We find a LEFT copy and RIGHT copy of the BMS algebra.

Boundary null fields are (holographically) causal and obey the antipodal map.

Reduction to single copy

Acting with finite superrotations $x^\pm \to X^\pm(x^\pm)$ on dS_2 we get

$$ds^2_{(0)} = -\frac{2\Delta^2 \partial_+ X^+ \partial_- X^-}{\cos(\Delta(X^+ + X^-)) + 1} dx^+ dx^- .$$

Asymptotic flatness at $\mathcal{I}^+_-$ imposes

$$\cos(\Delta(X^+ + X^-)) + 1 = 0, \qquad \text{at} \qquad T = \pm \frac{\pi}{2\Delta}.$$

This fixes

$$\begin{aligned} X^-(x) &= \frac{\pi}{\Delta} - X^+(\frac{\pi}{\Delta} - x) + \frac{2\pi}{\Delta} k, \qquad \forall x \\ X^+(x) &= X^+(x + \frac{2\pi}{\Delta}) + 2\pi \hat{k} \end{aligned}$$

where $k, \hat{k} \in \mathbb{Z}$. Also, one combination of supertranslations is pure gauge.

Final BMS_3 charge algebra

We obtain

- Supertranslation charge or supermomenta $\mathcal{P}_n$
- Superrotation charge or super Lorentz charge $\mathcal{J}_m$

They obey

$$
\begin{aligned}
i\{\mathcal{P}_m, \mathcal{P}_n\} &= 0, \\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\mathcal{P}_{m+n} + \frac{1}{4G} m(m^2-1)\delta_{m+n,0}, \\
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\mathcal{J}_{m+n}.
\end{aligned}
$$

at $\mathcal{I}^+$, $\mathcal{I}^-$, spatial infinity and in the bulk of spacetime thanks to $\omega = 0$.

Conclusion

- The memory effect is a fundamental and testable effect of GR. It points to the relevance of the supertranslation memory field.
- We built Poincaré vacua with supertranslation memory in $4d$ and $3d$. Supertranslations source superrotation charges.
- For stationary configurations, symplectic symmetries allow extend asymptotic charges to bulk charges
- We showed how the antipodal map property of asymptotically flat spacetimes can be explained in terms of causal propagation of null boundary fields
- Much remains to be understood.