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THE FIELD-THEORETICAL AND CANONICAL METHODS IN CONSTRUCTING CONSERVATION LAWS FOR PERTURBATIONS IN GRAVITY THEORIES

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"Recent Development in General Relativity"

Conference in memory of the late Joseph Katz

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♣ THEORY

♢ Katz, Bichak and Lynden-Bell conserved quantities.

♠ On the Komar anomaly [J. Katz, *CQG* 2 423 (1985)]:

- **Komar's superpotential**:

$$\mathcal{L}_H = -\frac{1}{16\pi}\sqrt{-g}R = -\frac{1}{16\pi}\hat{R} \quad \Longrightarrow \quad \mathcal{J}^{\mu\nu}_{Komar} \quad \Longrightarrow \quad E = \tfrac{1}{2}mc^2\,. \tag{1}$$

- **Katz's superpotential:**

$$\mathcal{L}_{Katz} = \mathcal{L}_H + \mathrm{div} \quad \Longrightarrow \quad \mathcal{J}^{\mu\nu}_{Katz} \quad \Longrightarrow \quad E = mc^2\,. \tag{2}$$

- $\mathcal{J}^{\mu}_{Katz}$ – **the covariantized Einstein's pseudotensor**;
- $\mathcal{J}^{\mu\nu}_{Katz}$ – **the covariantized Freud's superpotential**;

♠ The KBL bi-metric Lagrangian [KBL, *Phys. Rev. D* 55 5957 (1997)]:

$$\mathcal{L}_{KBL} = \mathcal{L}_H - \bar{\mathcal{L}}_H + \mathrm{div}\,. \tag{3}$$

- **Differential conservation law:**

$$\partial_\mu \mathcal{J}^{\mu}_{KBL} = 0 \quad \Longrightarrow \quad \mathcal{J}^{\mu}_{KBL} = \partial_\nu \mathcal{J}^{\mu\nu}_{KBL} \tag{4}$$

- $\mathcal{J}^{\mu}_{KBL}(\xi) \equiv \theta^{\mu}{}_{\nu}\xi^{\nu} + \sigma^{\mu\rho\sigma}\bar{\nabla}_\rho\xi_\sigma + z^\mu(\xi)$ – **the generalized Noether current**;
- $\theta^{\mu}{}_{\nu}$ – **the generalized energy-momentum**;
- $\sigma^{\mu\rho\sigma}$ – **the spin tensor**;
- $\mathcal{J}^{\mu\nu}_{KBL}$ – **the related superpotential**.

♢ The field-theoretical approach.

♠ The Deser Formulation [S. Deser, *GRG* 1, 9 (1970)]:

- **Deser's Lagrangian (Minkowski background, formally):**

$$\hat{g}^{\mu\nu} = \hat{\eta}^{\mu\nu} + \hat{h}^{\mu\nu} \qquad \Longrightarrow \qquad \mathcal{L}_D = -\frac{1}{16\pi}\left(\hat{R} - \hat{h}^{\mu\nu}\bar{R}_{\mu\nu} - \hat{\eta}^{\mu\nu}\bar{R}_{\mu\nu}\right) . \tag{5}$$

- **The field-theoretical equations (Einstein equations):**

$$\hat{G}^{L}_{\mu\nu} = 8\pi \hat{t}^{tot}_{\mu\nu} = 8\pi\left(\hat{t}^{g}_{\mu\nu} + \hat{t}^{m}_{\mu\nu}\right) . \tag{6}$$

- $\hat{G}^{L}_{\mu\nu}$ – **the linear in $\hat{h}^{\mu\nu}$ operator**;
- $\hat{t}^{tot}_{\mu\nu}$ – **the total energy-momentum density**;
- **Differential conservation law:**

$$\bar{\nabla}_\nu G^{\mu\nu}_L \equiv 0 \qquad \Longrightarrow \qquad \bar{\nabla}_\nu t^{\mu\nu}_{tot} = 0\,. \tag{7}$$

♠ The Grishchuk, Petrov and Popova generalization (curved backgrounds) [GPP, *Commun. Math. Phys.* **94**, 379 (1984)]:

$$\hat{g}^{\mu\nu} = \bar{\hat{g}}^{\mu\nu} + \hat{h}^{\mu\nu}, \quad \Phi^A = \bar{\Phi}^A + \phi^A \qquad \Longrightarrow \qquad \hat{\mathcal{L}} = -\frac{1}{16\pi}\hat{R} + \hat{\mathcal{L}}^M$$

$$\hat{\mathcal{L}}^{dyn}(\bar{g}_{\mu\nu}, \overline{\Phi}^A;\, \hat{h}^{\mu\nu}, \phi^B) = \hat{\mathcal{L}}(\overline{g} + h,\, \overline{\Phi} + \phi) - \hat{h}^{\mu\nu}\delta\overline{\hat{\mathcal{L}}}/\delta\overline{\hat{g}}^{\mu\nu} - \phi^A\delta\overline{\hat{\mathcal{L}}}/\delta\overline{\Phi}^A - \overline{\hat{\mathcal{L}}} \tag{8}$$

- **The field-theoretical equations (Einstein equations):**

$$\hat{G}^L_{\mu\nu} + \hat{\Phi}^L_{\mu\nu} = 8\pi \hat{t}^{tot}_{\mu\nu} \,. \tag{9}$$

- **The total energy-momentum tensor is not conserved:**

$$t^{\rm tot}_{\mu\nu} \equiv \frac{2}{\sqrt{\bar{g}}} \frac{\delta \mathcal{L}^{dyn}}{\delta \bar{g}^{\mu\nu}} \,, \qquad \bar{\nabla}_\nu t^{\mu\nu}_{\rm tot} \neq 0 \,. \tag{10}$$

♠ **Superpotentials and conserved currents:** [A.N.Petrov, *CQG*, 22, L83 (2005)]

- **Auxiliary linear Lagrangian** (generator of $\hat{G}^L_{\mu\nu}$):

$$\hat{\mathcal{L}}_1 = -\frac{1}{16\pi} \hat{h}^{\mu\nu} \frac{\delta \overline{\hat{\mathcal{L}}_H}}{\delta \bar{\hat{g}}^{\mu\nu}} \tag{11}$$

- **For arbitrary displacements along ξ^α Nœther procedure leads to:**

$$\partial_\mu \mathcal{J}^\mu_{ft} = 0 \qquad \Longrightarrow \qquad \mathcal{J}^\mu_{ft} = \partial_\nu \mathcal{J}^{\mu\nu}_{ft} \tag{12}$$

- $\mathcal{J}^\mu_{ft}(\xi) \equiv {}_{ft}\theta^\mu{}_\nu \xi^\nu + {}_{ft}z^\mu(\xi)$ – **the generalized f-t current** (no spin term!);
- ${}_{ft}\theta^\mu{}_\nu$ – **the generalized f-t energy-momentum**;
- $\mathcal{J}^{\mu\nu}_{ft}$ – **the related superpotential**.

♢ The generalized Belinfante procedure applied to the KBL model.

♠ Suppressing the spin term in current
[A. N. Petrov and J. Katz, *Proc. R. Soc. (L) A*, 458, 319 (2002)]:

- **The Belinfante generalized correction:**

$$s^{\mu\nu\rho} = -s^{\nu\mu\rho} = \sigma^{\rho[\mu\nu]} + \sigma^{\mu[\rho\nu]} - \sigma^{\nu[\rho\mu]}\,. \qquad (13)$$

- **The Belinfante corrected conservation law:**

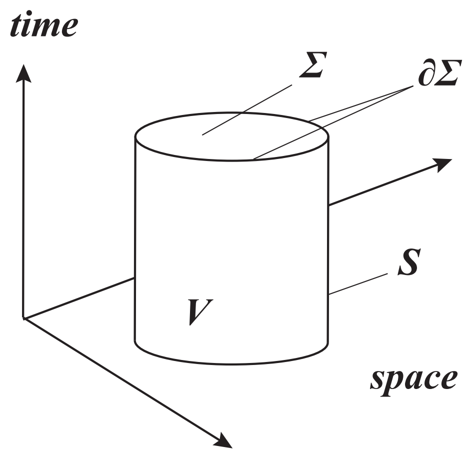
$$\mathcal{J}^{\mu}_{KBL} + \partial_\nu\left(s^{\mu\nu\rho}\xi_\rho\right) = \partial_\nu\left(\mathcal{J}^{\mu\nu}_{KBL} + s^{\mu\nu\rho}\xi_\rho\right) \quad \Longrightarrow \quad \mathcal{J}^{\mu}_{B} = \partial_\nu \mathcal{J}^{\mu\nu}_{B}, \qquad (14)$$

- $\mathcal{J}^{\mu}_{B}(\xi) \equiv {}_B\theta^{\mu}{}_{\nu}\xi^{\nu} + {}_Bz^{\mu}(\xi)$ – **the generalized Belinfante current** (no spin term!)

♠ Equivalence between the Belinfante corrected and f-t quantities:

- $\mathcal{J}^{\mu\nu}_{B}(\xi) \equiv \mathcal{J}^{\mu\nu}_{ft}(\xi)$ – **superpotentials** (automatically)
- ${}_B\theta^{\mu}{}_{\nu} \equiv {}_{ft}\theta^{\mu}$ – **energy-momentum** (using the field equations)

THUS IN GR, THE BELINFANTE PROCEDURE IS A BRIDGE BETWEEN F-T AND CANONICAL PROCEDURES



♢ <u>Integral conserved quantities:</u>

<u>TESTS FOR CHECKING VARIOUS APPROACHES.</u>

- **Integration of the differential conservation**, $\partial_\alpha \mathcal{J}^\alpha(\lambda) = 0$, **defines a conserved quantity:**

$$\mathcal{P}(\lambda) = \int_\Sigma \mathcal{J}^0(\lambda) d^3x\,; \tag{15}$$

- λ^α – **background Killing vectors**;
- **differential conservation**, $\mathcal{J}^\alpha(\lambda) = \partial_\alpha \mathcal{J}^{\alpha\beta}(\lambda)$, **defines a conserved quantity as a surface integral (charge):**

$$\mathcal{P}(\lambda) = \int_{\partial\Sigma} \mathcal{J}^{0k}(\lambda) ds_k \tag{16}$$

♢ <u>An arbitrary multidimensional metric theory.</u> Lagrangian - $\hat{\mathcal{L}}_G$.

♠ **Arbitrary theory, EGB theory, Lovelock theory:**
[N. Deruelle, J. Katz and S. Ogushi, *CQG*, <u>21</u>, 1971 (2004)],
[J. Katz and G. Livshits, *CQG*, <u>25</u>, 175024 (2008)].

- **The KBL ideology**
- **Different divergences**

♠ **Non-equivalence between the Belinfante and f-t quantities in general:**
[AP, *CQG*, <u>26</u>, 135010 (2009); *CQG*, <u>28</u>, 215021 (2011)],

$$\hat{\mathcal{L}}_1 = -\frac{1}{16\pi}\hat{h}^{\mu\nu}\frac{\delta\overline{\hat{\mathcal{L}}_G}}{\delta\overline{\hat{g}}^{\mu\nu}} \qquad \Longrightarrow \tag{17}$$

- **The F-T superpotential is linear in perturbation in any case.**
- **The Belinfante superpotential is not linear in general,** $\mathcal{J}_B^{\mu\nu} = \mathcal{J}_{KBL}^{\mu\nu} + s^{\mu\nu\rho}\xi_\rho$.

♠ **A family conserved quantities in KBL and B methods:**
[A. N. Petrov and R. R. Lompay, *GRG*, <u>45</u>, 545 (2013)],

♠ **Tests, like Sh-AdS mass, do not select a preferable approach.**

♠ **Interpretation of solutions.**

♣ SIMPLE APPLICATIONS

◊ KALUZA-KLEIN 3D BLACK HOLES IN EGB GRAVITY.

♠ **Maeda-Dadhich model:** [MD, *PRD*, 74, 021501(R) (2006); *PRD*, 75, 044007 (2007)]

- **The EGB action**

$$S = -\frac{1}{2\kappa_D}\int d^D x\sqrt{-g}\left[R - 2\Lambda_0 + \alpha\underbrace{\left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2\right)}_{L_{GB}}\right]. \tag{18}$$

- **The vacuum field equations:**

$$\mathcal{G}^{\mu}{}_{\nu} \equiv G^{\mu}{}_{\nu} + \alpha H^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\Lambda_0 = 0\,. \tag{19}$$

♠ **The Maeda-Dadhich assumptions.**

- **The EGB spacetime is to be locally homeomorphic to** $\mathcal{M}^d \times \mathcal{K}^{D-d}$.
- **The EGB metric:** $g_{\mu\nu} = \mathrm{diag}(g_{AB}, r_0^2\gamma_{ab}),\ A, B = 0, \cdots, d-1;\ a, b = d, \cdots, D-1.$
- g_{AB} **is an arbitrary Lorentz metric on** $\mathcal{M}^d$.
- γ_{ab} **is the unit metric on space of constant curvature** $\mathcal{K}^{D-d}$ **with** $k = 0, \pm 1$.
- r_0 **is a small scale of extra dimensions in** $\mathcal{K}^{D-d}$.

♠ The field equations are decomposed as

$$\mathcal{G}^A{}_B \equiv \left[1 + \frac{2k\alpha}{r_0^2}(D-d)(D-d-1)\right] {}_{(d)}G^A{}_B + \alpha\, {}_{(d)}H^A{}_B$$
$$+ \left[\Lambda_0 - \frac{k}{2r_0^2}(D-d)(D-d-1)\left(1 + \frac{k\alpha}{r_0^2}(D-d-2)(D-d-3)\right)\right] \delta^A{}_B = 0\,; \quad (20)$$
$$\mathcal{G}^a{}_b \equiv \delta^a{}_b \left\{ -\frac{{}_{(d)}R}{2} + \Lambda_0 - \frac{k}{2r_0^2}(D-d-1)(D-d-2) - \alpha\left[\frac{k}{r_0^2}(D-d-1)(D-d-2)\times\right.\right.$$
$$\left.\left.\times\left({}_{(d)}R + \frac{k}{2r_0^2}(D-d-3)(D-d-4)\right) + \frac{{}_{(d)}L_{GB}}{2}\right]\right\} = 0 \quad (21)$$

- '${}_{(d)}$' **means that a quantity is constructed with the use of** g_{AB}.

♠ To obtain interesting solutions MD consider the case, when $\mathcal{G}^A{}_B \equiv 0$

- **It is possible for** $d \le 4$ **only because then** ${}_{(d)}H_{\mu\nu} \equiv 0$;
- **to satisfy** $\mathcal{G}^A{}_B \equiv 0$ **one can choose only** $D \ge d+2$, $k=-1$ **and** $\Lambda_0 < 0$.

♠ We consider for simplicity $D = 6$ **and** $d = 3$, **then** $r_0^2 = 12\alpha = -3/\Lambda_0$.

♠ Then, only the field equation is

$$ {}_{(d)}R = 2\Lambda_0 \,, \tag{22} $$

• The static MD solution:

$$ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\phi, \quad f \equiv r^2/l^2 + q/r - \mu, \quad l^2 \equiv -3/\Lambda_0, \,. \tag{23} $$

μ and q are the constants of integration.

• For a comparison, the BTZ solution of the standard Einstein theory:

$$ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\phi \,, \qquad f \equiv -r^2 \Lambda_0 - \mu \,, \tag{24} $$

$r_+^2 = -\mu/\Lambda_0$ is the definition of the horizon with the background $\overline{f} = -r^2\Lambda_0$.

♠ BHs in the MD case are defined analogously.

• The MD horizon equation: $l^2 q + r_+(r_+^2 - l^2\mu) = 0$.

• The MD background: $\overline{f} \equiv r^2/l^2 + q/r - q/r_+$.

• The MD 'mass' parameter: $\tilde{\mu} = \mu - q/r_+$ with $r_+^2 = l^2\tilde{\mu}$.

♠ The MD interpretation.

- The Einstein tensor components in 3D, which are non-zero:

$$G_0^0 = G_1^1 = 1/l^2 - q/2r^3, \quad G_2^2 = 1/l^2 + q/r^3\,. \tag{25}$$

- Assuming that the Einstein equations are hold in 3D, MD conclude that they are not vacuum with redefined cosmological constant, $\Lambda = \Lambda_0/3 = -1/l^2$:

$$_{(3)}R_{AB} - \tfrac{1}{2}g_{AB\,(3)}R + g_{AB}\Lambda = \kappa_3 \mathcal{T}_{AB}\,. \tag{26}$$

- The 'matter' source $\mathcal{T}_{AB}$ with a zero trace $\mathcal{T}^A{}_A = 0$ has to exist; it is interpreted as created by the extra dimensions.

♠ We agree with this but think that it requires more fundamental basis.

♠ Calculation of masses of the objects.

- **Charge in all the approaches, KBL, field-theoretical and Belinfante corrected:**

$$\mathcal{P}(\xi) = \int_\Sigma d^{D-1}x\, \mathcal{J}^0(\xi) = \oint_{\partial\Sigma} dS_i\, \mathcal{J}^{0i}(\xi) \tag{27}$$

- **The timelike Killing vector:**

$$\xi^\alpha = (-1, \mathbf{0})\,. \tag{28}$$

♠ The BTZ case:

$$M = \oint_{r\to\infty} {}_E\mathcal{J}^{01} d\phi = \frac{\pi\mu}{\kappa_3}\,, \tag{29}$$

♠ The MD case:

- $D = 6$ **case in EGB gravity:**

$$M_6 = \oint_{\partial\Sigma} dx^{D-2}\sqrt{-\overline{g}_D}\, J^{01}_D = \underbrace{\oint_{r_0} dx^{D-d}\sqrt{-\overline{g}_{D-d}}}_{V_{r_0}} \oint_{r\to\infty} d\phi\sqrt{-\overline{g}_d}\, J^{01}_D \tag{30}$$

- $\mathbf{M_6 = 0}$ **because in all the cases**

$$\oint_{r\to\infty} d\phi\sqrt{-\overline{g}_d}\, J^{01}_6 \equiv 0$$

; **see also** [R.-G. Cai, L.-M. Cao and N. Ohta, *PRD*, 81, 024018 (2010).]

♠ Compactified extra dimensions:

- Because $\mathcal{I}_6^{01} \sim 1/\kappa_6$ one could set $\kappa_3 = \kappa_6/V_{r_0}$ - it is the Kaluza-Klein paradigma. Then

$$M_6 \sim \frac{1}{\kappa_3} \oint_{r\to\infty} d\phi \ldots \tag{31}$$

that is not a comfortable result.

- Therefore, we has to calculate:

$$M_3 = \oint_{r\to\infty} d\phi \sqrt{-\overline{g}_3}\, J_3^{01}, \tag{32}$$

where $J_3^{01} \sim 1/\kappa_3$ and $\kappa_3 = \kappa_6/V_{r_0}$.

- Then, in all the cases:

$$M_3 = \frac{\pi\tilde{\mu}}{\kappa_3}\,, \tag{33}$$

where $\tilde{\mu}$ is the reduced MD mass parameter.

♢ A POINT MASS IN GENERAL RELATIVITY.

♠ Coalescence of binary black holes:

[G. Schäfer, Post-Newtonian Methods: Analytic Results on the Binary Problem in book: *Mass and Motion in General Relativity*, 167–210 (Springer, 2011)]; [L. Blanchet, *LRR*, 17, 187 (2014)]; and others.

- **At an *initial* step the black holes are modeled by *point-like* particles presented by Dirac's δ-function.**
- **Then consequent post-Newtonian approximations are used; excellent mathematics, regularization, etc**
- **Extremely necessary for describing LIGO's and VIRGO's discovery!**

♠ An exact presentation.

- **The same physical reality is described in two mathematical techniques.**
- **Solution of interpretation problems.**

♢ The Schwarzschild solution as a field a point source in Minkowski space.

- **The Newtonian gravity:**

$$\varphi = \frac{m}{r} \implies \Delta\varphi = -4\pi m \delta(\vec{r}) \tag{34}$$

- **The Schwarzschild solution:**

$$ds^2 = (1 - r_g/r)\, c^2 dt^2 - (1 - r_g/r)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{35}$$

- **The Einstein equations:**

$$G_{\mu\nu} = \kappa T_{\mu\nu} \implies T_{\mu\nu} = ??? \qquad \text{(not satisfactory)}. \tag{36}$$

- **The field-theoretical form of the GR equations,**

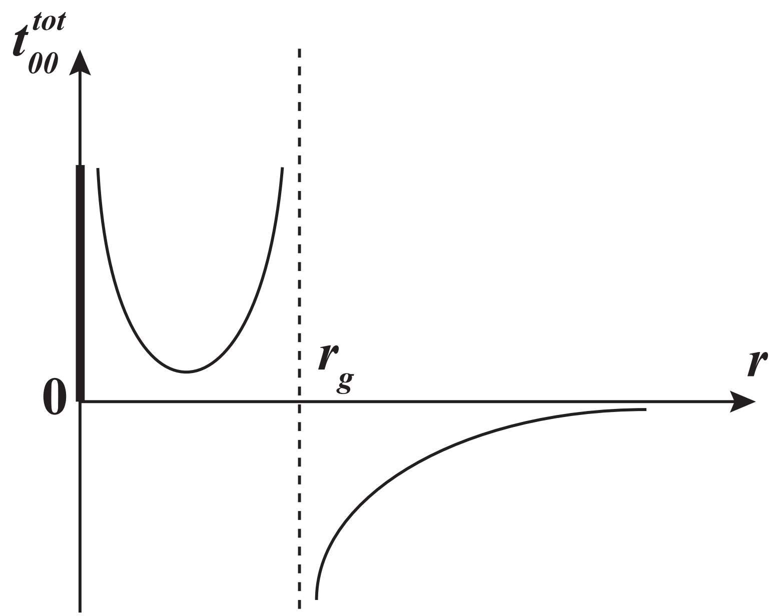
$$G^L_{\mu\nu} = t^{tot}_{\mu\nu}\,. \tag{37}$$

- **The backgound Minkowski space:**

$$d\bar{s}^2 = c^2 dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{38}$$

- **The field configuration:**

$$h^{00}_s = -\frac{r_g/r}{1 - r_g/r}\,, \qquad h^{11}_s = -\frac{r_g}{r}\,. \tag{39}$$



The energy density of the gravitational field and sources.

- The break looks as non-comfortable.
- The coordinate transformation, like $cdt \to cdt + f(r)dr$ applied to physical metric $g_{\mu\nu}$ and a consequent choose of the same background as Minkowski space

$$d\bar{s}^2 = c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{40}$$

changes the field configuration — it is interpreted as the gauge transformation.

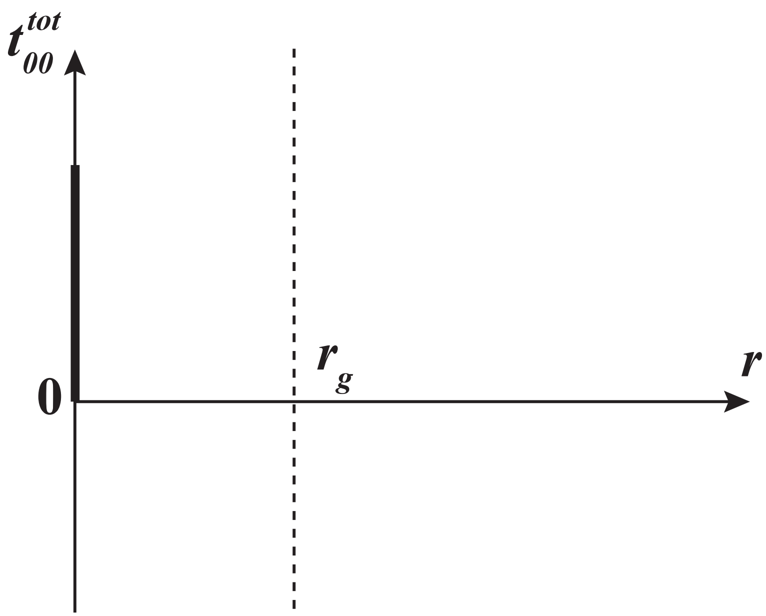
♠ It is necessary to find a more appropriate gauge fixing.

♠ Coordinates transformations for the Schwarzschild solution:

$$cdt \to cdt - f(r_g/r)dr. \tag{41}$$

♠ Required properties of the related field configurations:

- **the true singularity is placed at $r = 0$ by the δ-function ;**
- **regularity at the horizon;**
- **the field variables (perturbations) are asymptotically flat.**



♠ The Eddington-Finkelstein gauge fixing

- **The Schwarzschild solution:**

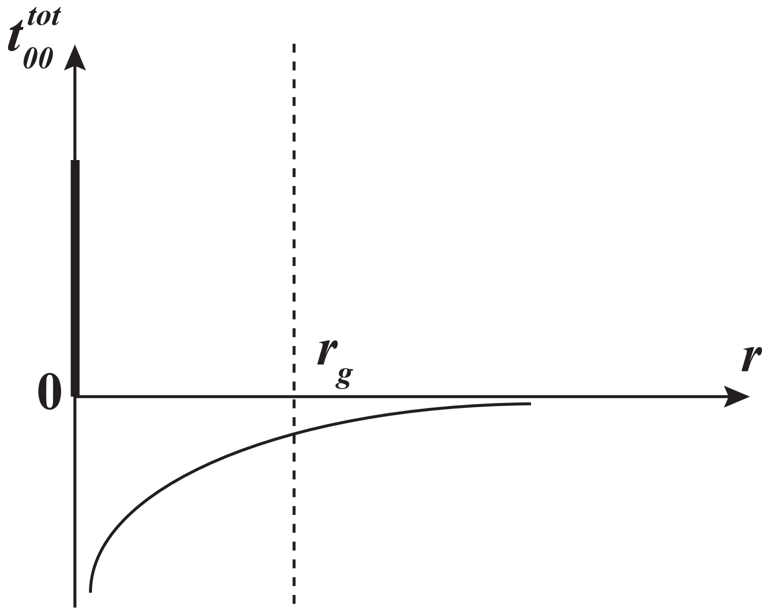
$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - 2c\,\frac{r_g}{r}\,dr dt - \left(1 + \frac{r_g}{r}\right) dr^2 - r^2 d^2\Omega. \tag{42}$$

- **in Minkowski space:** $d\bar{s}^2 = c^2 dt^2 - dr^2 - r^2 d^2\Omega.$
- **The field configuration:**

$$h_e^{00} = -\frac{r_g}{r}\,, \qquad h_e^{01} = \frac{r_g}{r}\,, \qquad h_e^{11} = -\frac{r_g}{r}\,. \tag{43}$$

- **Energy-momentum:**

$$t_{00}^{tot} = mc^2\delta(\mathbf{r})\,, \qquad t_{11}^{tot} = -mc^2\delta(\mathbf{r})\,, \qquad t_{AB}^{tot} = -\tfrac{1}{2}\bar{g}_{AB}\,mc^2\delta(\mathbf{r})\,. \tag{44}$$



♠ A general gauge fixing

- **The Schwarzschild solution:**

$$
\begin{aligned}
ds^2 = & -\left(1-\frac{r_g}{r}\right)c^2dt^2 + 2\left[\frac{r_g}{r} + \left(1-\frac{r_g}{r}\right)f\right]cdtdr \\
& + \left[\left(1+\frac{r_g}{r}\right) - 2\frac{r_g}{r}f - \left(1-\frac{r_g}{r}\right)f^2\right]dr^2 + r^2 d\Omega^2 \,.
\end{aligned} \tag{45}
$$

- **The field configuration:**

$$
\begin{aligned}
h_f^{00} &= -\frac{r_g}{r} + 2\frac{r_g}{r}f + \left(1-\frac{r_g}{r}\right)f^2 \,, \\
h_f^{01} &= \frac{r_g}{r} + \left(1-\frac{r_g}{r}\right)f \,, \\
h_f^{11} &= -\frac{r_g}{r} \,.
\end{aligned} \tag{46}
$$

♢ Announce of the monograph:

Petrov, Alexander N. / Kopeikin, Sergei M. / Lompay, Robert R. / Tekin, Bayram

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