Transgressions, boundary terms and Katz-like vectors in Lovelock gravity

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In memory of Joseph Katz

Collaboration with N. Deruelle and R. Olea
(to be submitted soon)

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Introduction

The variation of the Einstein-Hilbert (EH) action,

$$I_{\text{EH}}\left[g\right] = \kappa \int_{\mathcal{M}_4} d^4x\sqrt{-g}R\,, \quad \kappa = 1/16\pi G$$

gives on-shell the following boundary term (BT),

$$\delta I_{\text{EH}} = \kappa \int_{\mathcal{M}_4} d^4x\partial_\lambda\left(\sqrt{-g}V^\lambda\right) = \epsilon\kappa \int_{\partial\mathcal{M}_4} d^3x\sqrt{|h|}n_\lambda V^\lambda\,, \quad n_\mu n^\mu = \epsilon\,,$$

where

$$V^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\nu\mu} = \left(g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta}\right)\nabla\delta g_{\alpha\beta}\,.$$

Using $g^{\mu\nu} = \gamma^{\mu\nu} + n^\mu n^\nu$ we can write the boundary integral as

$$n_\lambda V^\lambda = n^\rho\gamma^{\mu\nu}\nabla_\mu\delta g_{\rho\nu} - \gamma^{\rho\nu}n^\mu\nabla_\mu\delta g_{\rho\nu} = (\ldots)^{\rho\nu}\,\delta g_{\rho\nu} + \underbrace{n^\rho\gamma^{\mu\nu}\partial_\mu\delta g_{\rho\nu}}_{\text{tangential}} + \underbrace{\gamma^{\rho\nu}n^\mu\partial_\mu\delta g_{\rho\nu}}_{\text{normal}}\,,$$

Here, one notice that it is **not consistent to fix both** the metric and its normal derivatives at the boundary.

The Lagrangian approache, this is solved by adding a suitable **boundary term (BT)**. However, the choice of the BT **is not unique**[1].

Before giving explicit examples, we notice that in Gaussian coordinates $ds^2 = N^2(r)\,dr^2 + h_{ij}(r,x)\,dx^i dx^j$, one has

$$\delta I_{\mathrm{EH}} = -2\kappa \int_{\partial\mathcal{M}_4} d^3x\sqrt{-h}\left(h^{ij}\delta K_{ij} + \frac{1}{2}K_{ij}\delta h^{ij}\right),$$

where $K_{ij} = \frac{1}{2N}\partial_r h_{ij}$ is the extrinsic curvature.

[1] York (1972); Gibbons, Hawking (1977); Katz (1985); Hawking, Horowitz (1996); Katz, Bičák, Lynden-Bell (1997).

Examples of boundary terms:

1) The Gibbons-Hawking-York (GHY) BT,

$$I_{\text{GHY}} = 2\kappa \int_{\partial \mathcal{M}_4} d^3x \sqrt{-h} K \,,$$

is such that the variation of $I_{\text{Dir,GHY}} \equiv I_{\text{EH}} + I_{\text{GHY}}$ is,

$$\delta I_{\text{Dir,GHY}} = \kappa \int_{\partial \mathcal{M}_4} d^3x \sqrt{|h|} b_{ij} \delta h^{ij}$$

with $b_{ij} \equiv K_{ij} - h_{ij} K$. Here b_{ij} can be identified[2] with the *canonical momenta* in Hamiltonian formalism.

[2] As mentioned e.g. in R. Olea and O. Mišković [arXiv:0706.4460], this relation is useful to understand Israel junction conditions for branes as a **discontinuity in the canonical momenta**. For the EGB case see: S.C. Davis [arXiv: hep-th/0208205]; E. Gravanis and S. Willison, [arXiv: hep-th/0209076]

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To obtain finite charges is still necessary to make **a background substraction**, which is made by Hawking-Howrowitz BT,

$$I_{\text{HH}} = 2\kappa \int_{\partial \mathcal{M}_4} d^3x \sqrt{-h} \left(K - \bar{K} \right) \,,$$

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2) Noticing that $\sqrt{-g}R = \sqrt{-g}G + \partial_\mu\left(\sqrt{-g}v^\mu\right)$ with $G = g^{\mu\nu}\left(\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right)$ and $v^\mu = g^{\nu\rho}\Gamma^\mu_{\nu\rho} - g^{\mu\nu}\Gamma^\rho_{\nu\rho}$ the simplest case would be to substract the divergence $\partial_\mu\left(\sqrt{-g}v^\mu\right)$ from I_{EH}.

3) The Katz BT,

$$
\begin{aligned}
I_{\mathrm{K}} &= \kappa \int_{\mathcal{M}_4} d^4x \partial_\mu\left(\sqrt{-g}k^\mu_{\mathrm{K}}\right) \\
k^\mu_{(1)} &= -\left(g^{\nu\rho}\Delta^\mu_{\nu\rho} - g^{\mu\nu}\Delta^\rho_{\nu\rho}\right), \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho}\,.
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I_{\rm K} &= \kappa\int_{\mathcal{M}_4} d^4x\partial_\mu\left(\sqrt{-g}k^\mu_{\rm K}\right) = 2\kappa\int_{\partial\mathcal{M}_4} d^3x\sqrt{-h}\left(K - \frac{1}{2}\left(h^{ij}+\bar{h}^{ij}\right)\bar{K}_{ij}\right), \\
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allows to define $I_{\rm Dir,K} \equiv I_{\rm EH} + I_{\rm K}$, whose variation

$$\delta I_{\rm Dir,K} = \kappa\int_{\partial\mathcal{M}_4} d^3x\sqrt{|h|}\left(b_{ij}\delta h^{ij} + c_{ij}\delta h^{ij}\right), \quad c_{ij} = \frac{1}{2}h_{ij}\left(h^{lk}+\bar{h}^{lk}\right)\bar{K}_{lk} - \bar{K}_{ij}$$

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The **regularized Dirichlet action**, in the context of Katz, Bicak, Lynden-Bell (1997) approach, is defined as $I_{\mathrm{reg,KBL}} \equiv I_{\mathrm{EH}} + I_{\mathrm{K}} - \bar{I}_{\mathrm{EH}}$.

On one hand, the GHY procedure was generalized Myers (1987) to the case of Lovelock gravity,

$$I_{\mathrm{L}} = \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p I^{(p)}\,, \quad I^{(p)} = \frac{(D-2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x\, \delta^{\nu_1 \dots \nu_{2p}}_{\mu_1 \dots \mu_{2p}} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots R^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}}\,,$$

For example, the Gibbons-Hawking-Myers (GHM) BT for the Gauss-Bonnet action $I^{(2)}$ is

$$I^{(2)}_{\mathrm{GHM}} = 4\,(D-4)! \int_{\mathcal{M}_D} d^d x \sqrt{-h} \delta^{j_1 j_2 j_3}_{i_1 i_2 i_3} K^{i_1}_{j_1} \left(\frac{1}{2} \mathcal{R}^{i_2 i_3}_{j_2 j_3}(h) - \frac{1}{3} K^{i_2}_{j_2} K^{i_3}_{j_3} \right)\,.$$

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There are two issues that allows to understand this generalization:

- The GHY procedure can be written in differential forms
- In that language the GHY term can be understood as a *dimensional continuation* (DC) the BT appearing in the Euler theorem.

On the other hand, up to know, a generalization of the Katz vector to Lovelock gravity **is not known** in the literature. In 2003, N. Deruelle, J. Katz and S. Ogushi constructed the vector,

$$k^{\mu}_{\text{DKO}} = 2k^{\mu}_{(2)}R + 8R^{\beta}_{\ \alpha}\Delta^{\mu\ \alpha}_{\ \beta} - 8R^{\mu}_{\ \alpha}\Delta^{\beta\ \alpha}_{\ \beta} - 4\Delta^{[\alpha\ \beta]}_{\ \gamma}g^{\gamma\rho}R^{\mu}_{\ \rho\alpha\beta}\,, \quad \Delta^{\mu\nu}_{\lambda} \equiv g^{\nu\alpha}\Delta^{\mu}_{\lambda\alpha}\,,$$

in such a way that gives the mass of the Boulware-Deser *bh* in Einstein-Gauss-Bonnet (EGB) theory. However,

- this vector does not solve the Dirichlet problem
- there is too much freedom to construct a vector from $(g_{\mu\nu}, \Delta^{\mu}_{\nu\rho})$.

In particular, we have noticed that is possible to add in many ways terms of the following type

$$k^{\mu}_{\text{DKO}} \rightarrow k^{\mu}_{\text{DKO}} + P^{\mu\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}\Delta^{\mu_1\mu_2}_{v_2}\nabla_{\nu_3}\Delta^{\mu_3\mu_4}_{\nu_4} + Q^{\mu\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}\Delta^{\mu_1\mu_2}_{v_2}\Delta^{\mu_3}_{\nu_3\lambda}\Delta^{\lambda\mu_4}_{\nu_4}\,,$$

in such a way that the value of the mass is not afected.

On this work we will show that a formulation of the Katz procedure in the language of **differential forms** (missing up to now) plus **topological arguments** gives a guideline to construct the vector

$$
\begin{aligned}
k^{\mu}_{(2)} &= 2k^{\mu}_{(1)}R - 4k^{\alpha}_{(1)}R^{\mu}_{\alpha} - 8\Delta^{[\alpha\mu]}_{\beta}R^{\beta}_{\alpha} - 4\Delta^{[\alpha\beta]}_{\gamma}g^{\gamma\rho}R^{\mu}{}_{\rho\alpha\beta} \\
&+ k^{\mu}_{(1)}\nabla_{\alpha}k^{\alpha}_{(1)} + 4\Delta^{[\alpha\beta]}_{\gamma}\nabla_{\alpha}\Delta^{[\mu\gamma]}_{\beta} + 8\Delta^{[\alpha\beta]}_{\beta}\nabla_{[\gamma}\Delta^{[\mu\gamma]}_{\alpha]} + 8\Delta^{[\mu\gamma]}_{\alpha}\nabla_{[\beta}\Delta^{[\alpha\beta]}_{\gamma]} \\
&- \frac{2}{3}\Big[-k^{\mu}_{(1)}\Delta^{[\beta\gamma]}_{\gamma}\Delta^{\alpha}_{\alpha\beta} + 2\Delta^{[\gamma\mu]}_{\beta}\Delta^{\rho}_{\rho\alpha}\Delta^{[\alpha\beta]}_{\gamma} + 2\Delta^{[\alpha\rho]}_{\gamma}\Delta^{\mu}_{\alpha\beta}\Delta^{[\beta\gamma]}_{\rho} \\
&+ k^{\gamma}_{(1)}\left(\Delta^{\mu}_{\gamma\beta}\Delta^{[\beta\alpha]}_{\alpha} + \Delta^{\alpha}_{\alpha\beta}\Delta^{[\beta\mu]}_{\gamma} - \Delta^{\alpha}_{\gamma\beta}\Delta^{[\beta\mu]}_{\alpha}\right) \\
&+ 2\Delta^{\gamma}_{\alpha\beta}\left(\Delta^{[\mu\rho]}_{\gamma}\Delta^{[\beta\alpha]}_{\rho} + \Delta^{[\mu\alpha]}_{\rho}\Delta^{[\beta\rho]}_{\gamma} + \Delta^{[\alpha\mu]}_{\gamma}\Delta^{[\beta\rho]}_{\rho} + \Delta^{[\rho\alpha]}_{\gamma}\Delta^{[\beta\mu]}_{\rho}\right)\Big] ,
\end{aligned}
$$

or

$$
k^{\mu}_{(2)} = -\delta^{\mu\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}\Delta^{\mu_1\mu_2}_{\nu_2}\left(R^{\mu_3\mu_4}_{\nu_3\nu_4} - \nabla_{\nu_3}\Delta^{\mu_3\mu_4}_{\nu_4} + \frac{2}{3}g_{\alpha\beta}\Delta^{[\mu_3\alpha]}_{\nu_3}\Delta^{[\beta\mu_4]}_{\nu_4}\right) ,
$$

which is *the one* allowing to solve the Dirichlet problem.

This will also give us a guideline to perform the *generalization* to a *generic Lovelock gravity*.

To make comparison with GHM procedure, in $D=5$ with Gaussian coordinates we have shown that:

$$I_{\rm K}^{(2)} = I_{\rm GHM}^{(2)} + \int_{\partial\mathcal{M}_5} d^4x\sqrt{-h}E$$

where,

$$\begin{aligned} E = {} & 4\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}\left\{-\frac{1}{2}\mathring{\nabla}_{i_2}\Delta^{[j_2j_3]}_{i_3} + K^{j_2}_{i_2}Z^{j_3}_{i_3} - \frac{1}{3}\left[K^{j_2}_{i_2}Z^{j_3}_{i_3} + Z^{j_2}_{i_2}\left(K^{j_3}_{i_3}+Z^{j_3}_{i_3}\right)\right] + \frac{1}{3}h_{l_2l_3}\mathring{\Delta}^{[j_2l_2]}_{i_2}\mathring{\Delta}^{[l_3j_3]}_{i_3}\right\} \\ & + 4\delta^{i_1i_2i_3}_{j_1j_2j_3}Z^{j_1}_{i_1}\left\{\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3} - \frac{1}{2}\mathring{\nabla}_{i_2}\Delta^{[j_2j_3]}_{i_3} + K^{j_2}_{i_2}Z^{j_3}_{i_3} - \frac{1}{3}\left(K^{j_2}_{i_2}+Z^{j_2}_{i_2}\right)\left(K^{j_3}_{i_3}+Z^{j_3}_{i_3}\right) + \frac{1}{3}h_{l_2l_3}\mathring{\Delta}^{[j_2l_2]}_{i_2}\mathring{\Delta}^{[l_3j_3]}_{i_3}\right\} \\ & - 2\delta^{i_1i_2i_3}_{j_1j_2j_3}\mathring{\Delta}^{j_1j_2}_{i_1}\left\{-2\epsilon\mathring{\nabla}_{i_2}K^{j_3}_{i_3} + \mathring{\nabla}_{i_2}\left(K^{j_3}_{i_3}+Z^{j_3}_{i_3}\right) - \frac{2}{3}h_{l_2l_3}\left(K^{l_2}_{i_2}+Z^{l_2}_{i_2}\right)\mathring{\Delta}^{[l_3j_3]}_{i_3}\right\} \end{aligned}$$

$$\mathring{\Delta}^j_{ik} = \mathring{\Gamma}^j_{ik} - \bar{\mathring{\Gamma}}^j_{ik}\,, \quad \mathring{\Delta}^{jl}_i \equiv h^{lk}\mathring{\Delta}^j_{ik}\,, \quad Z^j_i \equiv -\frac{1}{2}\left(h^{lj}+\bar{h}^{lj}\right)\bar{K}_{il}\,.$$

The variation of $I^{(2)}_{\rm Dir,K} \equiv I^{(2)} + I^{(2)}_{\rm K}$ has the form

$$\delta I^{(2)}_{\rm Dir,K} = \int_{\partial\mathcal{M}_D} d^4x\sqrt{|h|}\left(B_{ij}\delta h^{ij} + C_{ij}\delta h^{ij}\right)$$

Contents

EH gravity in the vielbein formulation

Vielbein and spin connection: $\left(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}\right) \rightarrow \left(e^A, \omega^A_{\ B}\right)$

$$e^A = e^A_\mu dx^\mu \,, \quad \omega^A_{\ B} = \omega^A_{\mu B} dx^\mu$$

The relation between both geometrical descriptions is ,

$$e^\mu_A e^\nu_B g_{\mu\nu} = \eta_{AB} \,, \quad \omega^A_{\mu B} = e^A_\alpha e^\gamma_B \Gamma^\alpha_{\mu\gamma} + e^A_\alpha \partial_\mu e^\alpha_B \,.$$

The curvature and torsion two-forms are defined by,

$$R^A_{\ B} = \frac{1}{2} R^A_{\mu\nu B} dx^\mu dx^\nu = d\omega^A_{\ B} + \omega^A_{\ C}\omega^C_{\ B} \,, \qquad R^A_{\mu\nu B} = e^A_\alpha e^\beta_B R^\alpha{}_{\beta\mu\nu} \,;$$

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To formulate GR, one needs to assume $T^A = 0$ (i.e., $\omega^A_{\mu B} = \omega^A_{\mu B}(e, \partial e)$) and introduce the Levi-Civita symbols ε_{ABCD}, $\varepsilon_{\mu\nu\alpha\beta}$.

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$$I_{\text{EH}}[g] = \kappa \int_{\mathcal{M}_4} d^4x \sqrt{-g} R = \frac{\kappa}{2} \int_{\mathcal{M}_4} \varepsilon_{ABCD} R^{AB} e^C e^D \equiv I_{\text{EH}}[e] \,,$$

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$$T^A = \frac{1}{2} T^A_{\mu\nu} dx^\mu dx^\nu = de^A + \omega^A_{\ B} e^B \,, \qquad T^A_{\mu\nu} = e^A_\lambda T^\lambda_{\mu\nu} \,.$$

To formulate GR, one needs to assume $T^A = 0$ (i.e., $\omega^A_{\mu B} = \omega^A_{\mu B}(e, \partial e)$) and introduce the Levi-Civita symbols ε_{ABCD}, $\varepsilon_{\mu\nu\alpha\beta}$. Using their properties (like, e.g., $\varepsilon_{ABCD} e^A_\mu e^B_\nu e^C_\alpha e^D_\beta = \sqrt{-g}\varepsilon_{\mu\nu\alpha\beta}$) one gets,

$$I_{\mathrm{EH}}[g] = \kappa \int_{\mathcal{M}_4} d^4x \sqrt{-g} R = \frac{\kappa}{2} \int_{\mathcal{M}_4} \varepsilon_{ABCD} R^{AB} e^C e^D \equiv I_{\mathrm{EH}}[e] \,,$$

$$\xrightarrow{T^A=0} \quad \delta I_{\mathrm{EH}} = \frac{\kappa}{2} \int_{\partial\mathcal{M}} d\left(\varepsilon_{ABCD} \delta\omega^{AB} e^C e^D\right) .$$

Euler theorem and the GHY term

The *Euler theorem* for $\mathcal{M}_2$

$$\int_{\mathcal{M}_2} \sqrt{-g} R d^2 x = -4\pi\chi\left(\mathcal{M}_2\right)$$

Euler theorem and the GHY term

The *Euler theorem* for $\mathcal{M}_2$ with $\partial\mathcal{M}_2 \neq 0$ and $ds^2 = N^2(r)\,dr^2 - h(r,t)\,dt^2$ reads

$$\int_{\mathcal{M}_2} \sqrt{-g} R d^2x = -4\pi\chi(\mathcal{M}_2) + \int_{\partial\mathcal{M}_2} \sqrt{-h} K dt\,, \quad K = \frac{1}{2N}\partial_r h$$

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To write this in differential forms it is necessary to introduce a reference manifold $\bar{\mathcal{M}}_2$ with metric $d\bar{s}^2 = N^2(r)\,dr^2 - \bar{h}(t)\,dt^2$, such that $\partial\mathcal{M}_2 = \partial\bar{\mathcal{M}}_2$. Then,

$$\int_{\mathcal{M}_2} \varepsilon_{AB} R^{AB} = -4\pi\chi(\mathcal{M}_2) + \int_{\partial\mathcal{M}_2} \varepsilon_{AB}\theta^{AB}\,,$$

where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ and $\bar{\omega}^{AB}$ is the reference spin connection.

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where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ and $\bar{\omega}^{AB}$ is the reference spin connection.One can recover the tensorial Euler theorem using

$$\begin{aligned} \omega^{1a} &= K^a = e^a_j K^j_i dx^i\,, \qquad K_{ij} = \frac{1}{2N}\partial_r h_{ij} \\ \bar{\omega}^{1a} &= \bar{K}^a = \bar{e}^a_j \bar{K}^j_i dx^i = 0\,, \quad \bar{K}_{ij} = 0 \\ \theta^{1a} &= K^a\,, \quad \theta^{ab} = \omega^{ab}\Big|_{\partial M} - \bar{\omega}^{ab}\Big|_{\partial M} = 0\,. \end{aligned} \tag{1}$$

where the indices are such that $a = 0$, $x^i = t$.

Dimensional continuation (DC) procedure:

$$\varepsilon_{AB}R^{AB} \quad \xrightarrow{\text{DC}} \quad \left\{ I_{\text{EH}} = \frac{\kappa}{2}\int_{\partial\mathcal{M}_4} \varepsilon_{ABCD}R^{AB}e^C e^D = \kappa\int_{\partial\mathcal{M}_4} d^4x\sqrt{-g}R \,, \right.$$

$$\varepsilon_{AB}\theta^{AB} \quad \xrightarrow{\text{DC}} \quad \left\{ \begin{aligned} I_{\text{GHY}} &= \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\beta_{(1)} = -\frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\theta^{AB}e^C e^D \\ &= \kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}K^a e^c e^d = 2\kappa\int_{\partial\mathcal{M}_4} d^3x\sqrt{-h}K \,, \end{aligned} \right.$$

where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ with ω^{AB} and $\bar{\omega}^{AB}$ being the spin connections of

$$ds^2 = N^2(r)\,dr^2 + h_{ij}(r,x)\,dx^i dx^j \quad ; \quad d\bar{s}^2 = \bar{N}^2(r)\,dr^2 + \bar{h}_{ij}(x)\,dx^i dx^j \,.$$

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Using

$$\delta I_{\text{EH}} = \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\delta\omega^{AB}e^C e^D = -\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\delta K^a e^b e^c\,,$$

Dimensional continuation (DC) procedure:

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$$\delta I_{\text{EH}} = \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\delta\omega^{AB}e^C e^D = -\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\delta K^a e^b e^c\,,$$

$$\Longrightarrow \qquad \delta I_{\text{Dir,GHY}} = -\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\delta K^a e^b e^c + \kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\left(\delta K^a e^b e^c + 2K^a e^b\delta e^c\right)$$

$$= 2\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}K^a e^b\delta e^c = \kappa\int_{\partial\mathcal{M}_4} d^3x\sqrt{|h|}b_{ij}\delta h^{ij}\,.$$

Similar constructions can be done in Lovelock gravity in $D = d + 1$ dimensions,

$$
\begin{aligned}
I_{\mathrm{L}} &= \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p I^{(p)} \,, \\
I^{(p)} &= \frac{(D-2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x \, \delta^{\nu_1 \cdots \nu_{2p}}_{\mu_1 \cdots \mu_{2p}} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots R^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}} \\
&= \int_{\mathcal{M}_D} \varepsilon_{A_1 \cdots A_D} R^{A_1 A_2} \cdots R^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \cdots e^{A_D} \,.
\end{aligned}
$$

In this case, the previous procedure to solve the Dirichlet problem was generalized by **Myers** (1987):

The so called *Gibbons-Hawking-Myers* (GHM) term $\beta_{(p)}$, which solves Dirichlet problem for each $I^{(p)}$, is contructed as the *DC* of the *BT* appearing in the $2p$-dimensional Euler theorem.

For example, the GHM term for the Gauss-Bonnet case,

$$\begin{aligned} I^{(2)} &= (D-2)! \int_{\mathcal{M}_D} d^D x \sqrt{-g} \left(R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right) \\ &= \int_{\mathcal{M}_D} \varepsilon_{A_1 \dots A_D} R^{A_1 A_2} R^{A_3 A_4} e^{A_5} \cdots e^{A_D} , \\ \delta\, I^{(2)} &= 2 \int_{\mathcal{M}_D} \varepsilon_{A_1 \dots A_D} \delta\omega^{A_1 A_2} R^{A_3 A_4} e^{A_5} \cdots e^{A_D} , \end{aligned}$$

is given by,

$$\begin{aligned} \beta_{(2)} &= -2\varepsilon_{A_1 \dots A_D} \theta^{A_1 A_2} \left(\mathcal{R}^{A_3 A_4}(h) + \frac{1}{3} \theta^{A_3}_{\ \ B} \theta^{B A_4} \right) e^{A_5} \cdots e^{A_D} \\ &= 4\,(D-4)! d^d x \sqrt{-h}\, \delta^{j_1 j_2 j_3}_{i_1 i_2 i_3} K^{i_1}_{j_1} \left(\frac{1}{2} \mathcal{R}^{i_2 i_3}_{j_2 j_3}(h) - \frac{1}{3} K^{i_2}_{j_2} K^{i_3}_{j_3} \right) , \end{aligned}$$

And this term can be regarded as the dimensional continuation of the *BT* in the 4-dimensional Euler theorem,

$$\int_{\mathcal{M}_4} \varepsilon_{ABCD} R^{AB} R^{CD} = 2\,(4\pi)^2 \chi(\mathcal{M}_4) + 2 \int_{\partial \mathcal{M}_4} \varepsilon_{ABCD} \theta^{AB} \left(\mathcal{R}^{CD}(h) + \frac{1}{3} \theta^{C}_{\ \ E} \theta^{ED} \right) e^C e^D ,$$

Contents

Katz procedure in vielbein formalism

The Katz *BT* can be written as,

$$I_{\mathrm{K}} = \kappa \int_{\partial \mathcal{M}_D} d^D x \partial_\mu \left(\sqrt{-g} k^\mu_{\mathrm{K}} \right) ,$$

where,

$$k^\mu_{\mathrm{K}} = - \left(g^{\nu\rho} \Delta^\mu_{\nu\rho} - g^{\mu\nu} \Delta^\rho_{\nu\rho} \right) ,$$
$$\Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho} ,$$

A smoth mapping $\sigma : \mathcal{M} \to \bar{\mathcal{M}}$ is chosen such that the **same coordinates** x^μ are used for the points $P \in M$ and $\bar{P} = \sigma(P) \in \bar{M}$. As a consequence:

- quantities like $g_{\mu\nu} + \bar{g}_{\mu\nu}$ transforms as a tensor,
- $\Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho}$ is a well defined tensor too.

Katz procedure in vielbein formalism

We can introduce vielbeins in both manifolds,

Manifold M	Manifold $\bar{M}$
$\forall\, P \in M,\ \exists\, x^\mu = x^\mu\left(y^A\right)$ such that,	$\forall\, \bar{P} = \sigma\left(P\right) \in \bar{M}\ \exists\, x^\mu = x^\mu\left(\bar{y}^A\right)$ such that,
$e^\mu_A e^\nu_B g_{\mu\nu} = \eta_{AB}$, with $e^\mu_A\left(P\right) = \frac{\partial x^\mu}{\partial y^A}\left(P\right)$.	$\bar{e}^\mu_A \bar{e}^\nu_B \bar{g}_{\mu\nu} = \eta_{AB}$, with $\bar{e}^\mu_A\left(\bar{P}\right) = \frac{\partial x^\mu}{\partial \bar{y}^A}\left(\bar{P}\right)$.

$$\bar{\omega}^{AB}_\mu = \bar{e}^A_\alpha \bar{e}^{B\gamma} \bar{\Gamma}^\alpha_{\mu\gamma} + \bar{e}^A_\alpha \partial_\mu \bar{e}^{B\alpha}\,, \tag{2}$$

$$\omega^{AB}_\mu = e^A_\alpha e^{B\gamma} \Gamma^\alpha_{\mu\gamma} + e^A_\alpha \partial_\mu e^{B\alpha}\,. \tag{3}$$

To solve the Dirichlet problem in the EH case one might naively try to construct the object

$$\beta^{(1)}\left(\omega, \bar{\omega}\right) = -\varepsilon_{A_1 \dots A_D} \theta^{A_1 A_2} e^{A_3} \cdots e^{A_D}\,,$$

similar to the one constructed in the GHY case, but where now $\theta = \omega - \bar{\omega}$ with $\bar{\omega}$ being **associated with the background metric,** instead of a product metric.

However, this does not allow us to transfrorm $\beta_{(1)}$ to the tensorial language. The reason is that the vielbeins are related as

$$e_{\mu}^{A}=\Omega_{B}^{A}\left(x\right)\bar{e}_{\mu}^{B},$$

where $\Omega_{B}^{A}\left(x\right)$ is not a Lorentz rotation, i.e., $\Omega^{T}\eta\Omega\neq\eta$ and this leads

$$\bar{\omega}_{\mu B}^{A}=\left(\Omega^{-1}\right)_{C}^{A}\Omega_{B}^{D}e_{\alpha}^{C}e_{D}^{\gamma}\bar{\Gamma}_{\mu\gamma}^{\alpha}+\left(\Omega^{-1}\right)_{C}^{A}e_{\alpha}^{C}\partial_{\mu}\left(e_{D}^{\alpha}\Omega_{B}^{D}\right).$$

Then it is not possible to use the identity $\varepsilon_{A_{1}\ldots A_{D}}e_{\mu_{1}}^{A_{1}}\cdots e_{\mu_{D}}^{A_{D}}=\sqrt{-g}\varepsilon_{\mu_{1}\ldots\mu_{D}}$, which is fundamental to make the translation to tensorial language.

Therefore, describing independently each manifold with differential forms is **not enough** to write the Katz divergence term in that language.

The new relevant geometrical object to introduce is the **"hybrid"** spin connection,

$$\tilde{\omega}_{\mu}^{AB} = \tilde{e}_{\alpha}^{A}\tilde{e}^{B\gamma}\tilde{\Gamma}_{\mu\gamma}^{\alpha} + \tilde{e}_{\alpha}^{A}\partial_{\mu}\tilde{e}^{B\alpha} = e_{\alpha}^{A}e^{B\gamma}\bar{\Gamma}_{\mu\gamma}^{\alpha} + e_{\alpha}^{A}\partial_{\mu}e^{B\alpha},$$

where $\tilde{e}_{\alpha}^{A}$ and $\tilde{\Gamma}_{\mu\gamma}^{\alpha}$ are the vielbein and the Christoffel connection of a **another**[3] **reference manifold** $\tilde{\mathcal{M}}_D$. Then, defining

$$I_{\mathrm{K}} = \frac{\kappa}{(D-2)!}\int_{\partial\mathcal{M}_D}\beta^{(1)}, \quad \beta^{(1)} = \varepsilon_{A_1\cdots A_D}\theta^{A_1A_2}e^{A_3}\cdots e^{A_D}$$

$$\theta_{\mu}^{AB} \equiv \theta_{\mu}^{AB}(\omega,\tilde{\omega}) = \omega_{\mu}^{AB} - \tilde{\omega}_{\mu}^{AB} = e_{\alpha}^{A}e^{B\gamma}\left(\Gamma_{\mu\gamma}^{\alpha} - \bar{\Gamma}_{\mu\gamma}^{\alpha}\right) = e_{\alpha}^{A}e^{B\gamma}\Delta_{\mu\gamma}^{\alpha},$$

we have shown that the following relation holds,

$$\frac{1}{(D-2)!}d\beta^{(1)} = d^Dx\partial_{\mu}\left(\sqrt{-g}k_{\mathrm{K}}^{\mu}\right),$$

i.e., the Katz BT can also be written in differential forms .

[3]We also choose a smooth mapping allowing us to use the same coordinates x^{μ} for each point $P \in \mathcal{M}_D$ and $\tilde{P} \in \tilde{\mathcal{M}}_D$.

We notice that $\tilde{\mathcal{M}}_D$ **is not a new independent manifold**, because $\tilde{\omega}_\mu^{AB} = \tilde{\omega}_\mu^{AB}\left(\bar{\Gamma}, e\right)$. Given $\mathcal{M}_D$ and $\bar{\mathcal{M}}_D$ it is always possible to determine the geometry of $\tilde{\mathcal{M}}_D$.

For example, for$\mathcal{M}_D$ and $\bar{\mathcal{M}}_D$ being static spherically symmetric spacetimes,

$$ds^2 = -f^2\left(r\right)dt^2 + \frac{1}{g^2\left(r\right)}dr^2 + r^2 d\Omega_{D-2}^2\,, \quad d\bar{s}^2 = -\bar{f}^2\left(r\right)dt^2 + \frac{1}{\bar{g}^2\left(r\right)}dr^2 + r^2 d\Omega_{D-2}^2\,,$$

we obtain,

$$d\tilde{s}^2 = -\tilde{f}^2\left(r\right)dt^2 + \frac{1}{\tilde{g}^2\left(r\right)}dr^2 + r^2 d\Omega_{D-2}^2\,,$$

$$\tilde{g}\left(r\right) = \frac{\bar{g}^2\left(r\right)}{g\left(r\right)}\,,\ \tilde{f}'\left(r\right) = \frac{g^2\left(r\right)f\left(r\right)\bar{f}'\left(r\right)}{\bar{g}^2\left(r\right)\bar{f}\left(r\right)}\,,$$

$$\tilde{\Gamma}^t_{\mu r} = \frac{f\left(r\right)g^2\left(r\right)}{\bar{f}\left(r\right)\bar{g}^2\left(r\right)}\bar{\Gamma}^t_{\mu r}\,,\ \tilde{\Gamma}^r_{ij} = \frac{\bar{g}^2\left(r\right)}{g^2\left(r\right)}\bar{\Gamma}^r_{ij}\,,\ \tilde{\Gamma}^i_{ik} = \bar{\Gamma}^i_{ik} = \Gamma^i_{ik}\,.$$

Although this spin connection has vanishing torsion, it does not satisfy the metricity condition, and thus, $\tilde{\Gamma}$ is not a Christoffel symbol.

This proposal **solves** the Dirichlet problem in diff. forms language.

For example, in $D = 4$ the variation of $I_{\mathrm{K}} = \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\beta^{(1)}$ is

$$\begin{aligned}\delta I_{\mathrm{K}} &= \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\delta\left[-\varepsilon_{ABCD}\theta^{AB}e^{C}e^{D}\right] \\ &= -\frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\delta\omega^{AB}e^{C}e^{D} + \frac{\kappa}{2}\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\delta\tilde{\omega}^{AB}e^{C}e^{D} - \kappa\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\theta^{AB}e^{C}\delta e^{D}\end{aligned}$$

where we have used $\theta^{AB} = \omega^{AB} - \tilde{\omega}^{AB}$. Thus, δI_{K} does cancel the *BT* comming from the variation of the EH action,

$$\delta I_{\mathrm{EH}} = \frac{\kappa}{2}\int_{\mathcal{M}_4}\varepsilon_{A_1\cdots A_D}\delta\omega^{A_1A_2}e^{A_3}\dots e^{A_D}$$

and the rest vanishes for the Dirichlet condition $\delta e^{A}_{\mu}\Big|_{\partial\mathcal{M}_4} = 0$. Indeed,

$$d\left(\varepsilon_{ABCD}\delta\tilde{\omega}^{AB}e^{C}e^{D}\right) = d\left[\varepsilon_{ABCD}f_{E}^{\nu,AB}\delta e^{E}_{\nu}e^{C}e^{D} - d\left(\varepsilon_{ABCD}e^{A}_{\nu}e^{C}e^{D}\right)\delta e^{B\nu}\right],$$

$$f_{E}^{\nu,AB}\left(e,\partial e,\bar{\Gamma}\right) = \left(\delta^{A}_{E}\delta^{\nu}_{\alpha}e^{B\gamma}\bar{\Gamma}^{\alpha}_{\mu\gamma} + \delta^{B}_{E}\delta^{\nu}_{\beta}e^{A}_{\alpha}g^{\beta\gamma}\bar{\Gamma}^{\alpha}_{\mu\gamma} + \delta^{A}_{E}\delta^{\nu}_{\alpha}\partial_{\mu}e^{B\alpha}\right)dx^{\mu}.$$

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$$f_E^{\nu,AB}\left(e,\partial e,\bar{\Gamma}\right) = \left(\delta^A_E\delta^\nu_\alpha e^{B\gamma}\bar{\Gamma}^\alpha_{\mu\gamma}+\delta^B_E\delta^\nu_\beta e^A_\alpha g^{\beta\gamma}\bar{\Gamma}^\alpha_{\mu\gamma}+\delta^A_E\delta^\nu_\alpha\partial_\mu e^{B\alpha}\right)dx^\mu\,.$$

To show consistency with tensorial formalism, the variation of $I_{\text{Dir,K}} = I_{\text{EH}} + I_{\text{K}}$,

$$\delta I_{\text{Dir,K}} = \frac{\kappa}{2}\int_{\partial\mathcal{M}_4} \varepsilon_{ABCD}\delta\tilde{\omega}^{AB}e^C e^D - \kappa\int_{\partial\mathcal{M}_4} \varepsilon_{ABCD}\theta^{AB}e^C\delta e^D$$

can be written in Gauss coordinates and then, transformed to tensorial language. This way, we get

$$\begin{aligned}\delta I_{\text{Dir,K}} &= -\frac{\epsilon\kappa}{2}\int_{\partial\mathcal{M}_D} \varepsilon_{abc}\left(\delta\tilde{\omega}^{\text{na}} - \delta\tilde{\omega}^{\text{an}}\right)e^b e^c - \kappa\int_{\partial\mathcal{M}_D} \epsilon\varepsilon_{abc}\left(2\epsilon K^a + \left(\tilde{\omega}^{\text{na}} - \tilde{\omega}^{\text{an}}\right)\right)e^b\delta e^c \\ &= 2\epsilon\kappa\int_{\partial\mathcal{M}_D} d^3x\sqrt{|h|}\delta h^{ij}\left(-\frac{1}{2}\bar{K}_{ij} + \frac{1}{4}h_{jl}\left(h^{lk} + \bar{h}^{lk}\right)\bar{K}_{ik}\right) \\ &\quad - \epsilon\kappa\int_{\partial\mathcal{M}_D} d^3x\sqrt{|h|}\left[Kh_{ij} - K_{ij} - \frac{1}{2}h_{ij}\left(h^{kl} + \bar{h}^{kl}\right)\bar{K}_{kl} + \frac{1}{2}h_{jl}\left(h^{lk} + \bar{h}^{lk}\right)\bar{K}_{ik}\right]\delta h^{ij} \\ &= \epsilon\kappa\int_{\partial\mathcal{M}_4} d^3x\sqrt{|h|}\left(b_{ij}\delta h^{ij} + c_{ij}\delta h^{ij}\right),\end{aligned}$$

$$b_{ij} = K_{ij} - h_{ij}K, \quad c_{ij} = \frac{1}{2}h_{ij}\left(h^{lk} + \bar{h}^{lk}\right)\bar{K}_{lk} - \bar{K}_{ij}$$

Contents

Katz-like vectors in Lovelock theory

The generalization for a Lovelock term $I^{(p)}$ requires to consider a more general version of the Euler theorem:

The Chern-Weil theorem (CWth)

Let $A = A_{\mu}^{M} T_{M} dx^{\mu}$ and $\bar{A} = \bar{A}_{\mu}^{M} T_{M} dx^{\mu}$ be two one-forms gauge connection and F, $\bar{F}$ the corresponging strenght fields (e.g., $F = dA + AA$). Then,

$$\begin{aligned} \langle F^{p} \rangle - \langle \bar{F}^{p} \rangle &= dT^{(2p-1)}(A, \bar{A}) \,, \\ T^{(2p-1)}(A, \bar{A}) &= p \int_{0}^{1} dt \left\langle \theta F_{t}^{p-1} \right\rangle , \quad \theta = A - \bar{A} \,, \end{aligned}$$

where F_t is the strenght field of the conection $A_t = \bar{A} + t(A - \bar{A})$ which interpolates between $\bar{A}$ and A. Here, $T^{(2p-1)}(A, \bar{A})$ is called **transgression form**. Its variation is given by,

$$\delta T^{(2p-1)}(A, \bar{A}) = p \left\langle F^{p-1} \delta A \right\rangle - p \left\langle \bar{F}^{p-1} \delta \bar{A} \right\rangle - p(p-1)\, d \int_{0}^{1} dt \left\langle \theta F_{t}^{p-2} \delta A_{t} \right\rangle .$$

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The Chern-Weil theorem (CWth)

Let $A = A_{\mu}^{M} T_M dx^{\mu}$ and $\bar{A} = \bar{A}_{\mu}^{M} T_M dx^{\mu}$ be two one-forms gauge connection and $F, \bar{F}$ the corresponging strenght fields (e.g., $F = dA + AA$). Then,

$$\begin{aligned} \langle F^p \rangle - \langle \bar{F}^p \rangle &= dT^{(2p-1)}(A, \bar{A}) , \\ T^{(2p-1)}(A, \bar{A}) &= p \int_0^1 dt \left\langle \theta F_t^{p-1} \right\rangle , \quad \theta = A - \bar{A} , \end{aligned}$$

where F_t is the strenght field of the conection $A_t = \bar{A} + t(A - \bar{A})$ which interpolates between $\bar{A}$ and A. Here, $T^{(2p-1)}(A, \bar{A})$ is called **transgression form**. Its variation is given by,

$$\delta T^{(2p-1)}(A, \bar{A}) = p \left\langle F^{p-1} \delta A \right\rangle - p \left\langle \bar{F}^{p-1} \delta \bar{A} \right\rangle - p(p-1) d \int_0^1 dt \left\langle \theta F_t^{p-2} \delta A_t \right\rangle .$$

Consider the case where $T_M = J_{AB}$ are Lorentz generators, so that

$$\begin{aligned}
\varepsilon_{A_1\ldots A_{2p}} &= \left\langle J_{A_1A_2}\cdots J_{A_{2p-1}A_{2p}}\right\rangle \\
A &= \frac{1}{2}\omega^{AB}J_{AB}\,, \quad \bar{A} = \frac{1}{2}\bar{\omega}^{AB}J_{AB}\,, \quad \tilde{A} = \frac{1}{2}\tilde{\omega}^{AB}J_{AB}\,, \\
F &= \frac{1}{2}R^{AB}J_{AB}\,, \quad \bar{F} = \frac{1}{2}\bar{R}^{AB}J_{AB}\,, \quad \tilde{F} = \frac{1}{2}\tilde{R}^{AB}J_{AB}\,,
\end{aligned}$$

where $\bar{\omega}$ and $\tilde{\omega}$ correspond to the *product metric* and *hybrid* cases.

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\end{aligned}$$

where $\bar{\omega}$ and $\tilde{\omega}$ correspond to the *product metric* and *hybrid* cases.

- For $(\omega, \bar{\omega})$ the CWth reproduces the $2p$-dimensional Euler theorem with the BT $T^{(2p-1)}(\omega, \bar{\omega})$. The GHM term is $I^{(p)}_{\text{GHM}} = \frac{\alpha_p}{(D-2)!} \int_{\mathcal{M}_D} d\beta_{(p)}$ where $\beta_{(p)}$ is the DC of $T^{(2p-1)}(\omega, \bar{\omega})$.

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- For $(\omega, \tilde{\omega})$ the CWth give us the transgresion form

$$
\begin{aligned}
T^{(2p-1)}(\omega, \tilde{\omega}) = -p\int_0^1 dt\varepsilon_{A_1\ldots A_{2p}}\theta^{A_1A_2}\left(R^{A_3A_4} + tD\theta^{A_3A_4} + t^2\eta_{B_1C_1}\theta^{[A_3B_1]}\theta^{[C_1A_4]}\right)\times\cdots \\
\cdots\times\left(R^{A_{2p-1}A_{2p}} + tD\theta^{A_{2p-1}A_{2p}} + t^2\eta_{B_pC_p}\theta^{\left[A_{2p-1}B_p\right]}\theta^{\left[C_pA_{2p}\right]}\right),
\end{aligned}
$$

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F &= \frac{1}{2}R^{AB}J_{AB}\,, \quad \bar{F} = \frac{1}{2}\bar{R}^{AB}J_{AB}\,, \quad \tilde{F} = \frac{1}{2}\tilde{R}^{AB}J_{AB}\,,
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- For $(\omega,\bar{\omega})$ the CWth reproduces the $2p$-dimensional Euler theorem with the BT $T^{(2p-1)}(\omega,\bar{\omega})$. The GHM term is $I^{(p)}_{\mathrm{GHM}} = \frac{\alpha_p}{(D-2)!}\int_{\mathcal{M}_D} d\beta_{(p)}$ where $\beta_{(p)}$ is the DC of $T^{(2p-1)}(\omega,\bar{\omega})$.

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$$\begin{aligned}
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\cdots\times\left(R^{A_{2p-1}A_{2p}} + tD\theta^{A_{2p-1}A_{2p}} + t^2\eta_{B_pC_p}\theta^{[A_{2p-1}B_p]}\theta^{[C_pA_{2p}]}\right)\,,
\end{aligned}$$

  and we have shown the Katz-like boundary term is $I^{(p)}_{\mathrm{K}} = \frac{\alpha_p}{(D-2)!}\int_{\mathcal{M}_D} d\beta^{(p)}$ with $\beta^{(p)}$ being the DC of $T^{(2p-1)}(\omega,\tilde{\omega})$.

In particular, for $p=2$,

$$T^{(3)}\left(\omega,\tilde{\omega}\right)=2\varepsilon_{ABCD}\theta^{AB}\left(R^{CD}-\frac{1}{2}D\theta^{CD}+\frac{1}{3}\eta_{EF}\theta^{[CE]}\theta^{[FD]}\right)$$

Then we define the DC of $T^{(3)}\left(\omega,\tilde{\omega}\right)$ as,

$$\beta^{(2)}=-2\varepsilon_{A_1\ldots A_D}\theta^{A_1A_2}\left(R^{A_3A_4}-\frac{1}{2}D^{(W)}\theta^{A_3A_4}+\frac{1}{3}\eta_{EF}\theta^{[A_3E]}\theta^{[FA_4]}\right)e^{A_5}\ldots e^{A_D}\,.$$

We have shown that $I_K^{(2)}=\frac{\alpha_2}{(D-2)!}\int_{\mathcal{M}_D}d\beta^{(2)}$ solves the Dirichlet problem and that the following relation holds

$$\frac{1}{(D-4)!}d\beta^{(2)}=\partial_\mu\left(\sqrt{-g}k^\mu_{(2)}\right)d^Dx\,,$$

$$k^\mu_{(2)}=-\delta^{\mu\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}\Delta^{\mu_1\mu_2}_{\nu_2}\left(R^{\mu_3\mu_4}_{\nu_3\nu_4}-\nabla_{\nu_3}\Delta^{\mu_3\mu_4}_{\nu_4}+\frac{2}{3}g_{\alpha\beta}\Delta^{[\mu_3\alpha]}_{\nu_3}\Delta^{[\beta\mu_4]}_{\nu_4}\right)\,,$$

i.e., the term $\beta^{(2)}$, constructed from the CWth, give us the form that the Katz-like vector for the GB term must have.

Therefore, in the KLB approach, the regularized Dirichlet Lovelock action we are proposing is: $I_{\mathrm{L,reg\text{-}KBL}} = I_{\mathrm{L}} + I_{\mathrm{K}} - \bar{I}_{\mathrm{L}}$, where

$$I_{\mathrm{L}} = \sum_{p=0}^{[D/2]} \alpha_p I^{(p)} ,$$

$$\begin{aligned} I^{(p)} &= \int_{\mathcal{M}_D} \varepsilon_{A_1 \cdots A_D} R^{A_1 A_2} \cdots R^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \cdots e^{A_D} \\ &= \frac{(D-2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x \, \delta^{\nu_1 \dots \nu_{2p}}_{\mu_1 \dots \mu_{2p}} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots R^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}} , \end{aligned}$$

$$I_{\mathrm{K}} = \int_{\mathcal{M}_D} \sum_{p=0}^{[D/2]} \frac{\alpha_p}{(D-2p)!} d\beta^{(p)} = \int_{\mathcal{M}_D} \sum_{p=0}^{[D/2]} \alpha_p \partial_\mu \left(\sqrt{-g} k^\mu_{(p)} \right) d^D x ,$$

$$\begin{aligned} \beta^{(p)} &= -p \int_0^1 dt \varepsilon_{A_1 \dots A_D} \theta^{A_1 A_2} \left(R^{A_3 A_4} + tD\theta^{A_3 A_4} + t^2 \eta_{B_1 C_1} \theta^{[A_3 B_1]} \theta^{[C_1 A_4]} \right) \times \cdots \\ &\cdots \times \left(R^{A_{2p-1} A_{2p}} + tD\theta^{A_{2p-1} A_{2p}} + t^2 \eta_{B_p C_p} \theta^{[A_{2p-1} B_p]} \theta^{[C_p A_{2p}]} \right) e^{A_{2p+1}} \cdots e^{A_D} , \end{aligned}$$

$$\begin{aligned} k^\mu_{(p)} &= -p \int_0^1 dt \delta^{\mu \mu_2 \dots \mu_{2p}}_{\nu_1 \dots \nu_{2p}} \Delta^{\nu_1 \nu_2}_{\mu_2} \left(\frac{1}{2} R^{\nu_3 \nu_4}_{\mu_3 \mu_4} + t \nabla_{\mu_3} \Delta^{\nu_3 \nu_4}_{\mu_4} + t^2 g_{\alpha_1 \beta_1} \Delta^{[\nu_3 \alpha_1]}_{\mu_3} \Delta^{[\beta_1 \nu_4]}_{\mu_4} \right) \times \\ &\cdots \times \left(\frac{1}{2} R^{\nu_{2p-1} \nu_{2p}}_{\mu_{2p-1} \mu_{2p}} + t \nabla_{\mu_{2p-1}} \Delta^{\nu_{2p-1} \nu_{2p}}_{\mu_{2p}} + t^2 g_{\alpha_p \beta_p} \Delta^{[\nu_{2p-1} \alpha_p]}_{\mu_{2p-1}} \Delta^{[\beta_p \nu_{2p}]}_{\mu_{2p}} \right) . \end{aligned}$$

Contents

Conclusions

- The missing formulation in differential forms of the KLB procedure has been given.
- In particular, we have shown that there is a unique generalization of Katz vector for EGB and Lovelock gravities.
- The new *BTs* we have obtained are different from GHM BT.
- In the case of EGB action our vector is not DKO vector, but it gives the same mass for the Boulware-Deser *bh*.

Future directions

- It remains to check if it also gives the correct conserved charges for other cases like, e.g., in Pure Lovelock gravity[4].
- We also want to compare our results with other procedures (the intrinsic and extrinsic regularization)
- In particular it would be interesting to find a relation with the **A. Petrov approach**.
- Then, it would be interesting to find an explicit proof that this proposal gives the correct conserved charges for *families of solutions*.
- To start, we will see if is possible to give a proof for locally asymptotically AdS solutions in Lovelock gravities with AdS vacuum (using Fefferman-Graham coordinates).

[4] R. G. Cai and N. Ohta [hep-th/0604088]; N. Dadhich, J. M. Pons and K. Prabhu [arXiv:1201.4994].

Future directions

- On the other hand, a recent application of the Katz procedure was made in the literature[5] for EH gravity coupled with scalar and Maxwell fields. Thus, it would be interesting to investigate also the consequences of adding extra fields.
- Finally, it would also be interesting to investigate the consequences of these results with a possible generalization of the Israel Junction conditions for the Lovelock case.[6].

[5] A. Anabalón, N. Deruelle and F. Julié [arXiv:1606.05849].

[6] See e.g.: O. Miskovic and R. Olea, [arXiv:0706.4460]; S.C. Davis [arXiv: hep-th/0208205]; E. Gravanis and S. Willison, [arXiv: hep-th/0209076].

Thanks