Transgressions, boundary terms and Katz-like vectors in Lovelock gravity

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In memory of Joseph Katz

Collaboration with N. Deruelle and R. Olea

(to be submitted soon)
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## Contents

1. Introduction

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The variation of the Einstein-Hilbert (EH) action,

\[ I_{EH} [g] = \kappa \int_{\mathcal{M}_4} d^4 x \sqrt{-g} R, \quad \kappa = 1/16\pi G \]

gives on-shell the following boundary term (BT),

\[ \delta I_{EH} = \kappa \int_{\mathcal{M}_4} d^4 x \partial_\lambda \left( \sqrt{-g} V^\lambda \right) = \epsilon \kappa \int_{\partial \mathcal{M}_4} d^3 x \sqrt{|h|} n_\lambda V^\lambda , \quad n_\mu n^\mu = \epsilon , \]

where

\[ V^\lambda = g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^\nu_{\nu\mu} = \left( g^{\lambda\alpha} g^{\nu\beta} - g^{\lambda\nu} g^{\alpha\beta} \right) \nabla \delta g_{\alpha\beta} . \]

Using \( g^{\mu\nu} = \gamma^{\mu\nu} + n^\mu n^\nu \) we can write the boundary integral as

\[ n_\lambda V^\lambda = n_\rho \gamma^{\mu\nu} \nabla_\mu \delta g_{\rho\nu} - \gamma^{\rho\nu} n^\mu \nabla_\mu \delta g_{\rho\nu} = (\ldots)^{\rho\nu} \delta g_{\rho\nu} + n_\rho \gamma^{\mu\nu} \partial_\mu \delta g_{\rho\nu} + \gamma^{\rho\nu} n^\mu \partial_\mu \delta g_{\rho\nu} , \]

\( \text{tangential} \quad \text{normal} \)
Here, one notice that it is **not consistent to fix both** the metric and its normal derivatives at the boundary.

The Lagrangian approach, this is solved by adding a suitable **boundary term** (BT). However, the choice of the BT is not unique\(^1\).

Before giving explicit examples, we notice that in Gaussian coordinates \(ds^2 = N^2(r) dr^2 + h_{ij}(r,x) dx^i dx^j\), one has

\[
\delta I_{EH} = -2\kappa \int_{\partial\mathcal{M}_4} d^3x \sqrt{-h} \left( h^{ij} \delta K_{ij} + \frac{1}{2} K_{ij} \delta h^{ij} \right),
\]

where \(K_{ij} = \frac{1}{2N}\partial_r h_{ij}\) is the extrinsic curvature.

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\(^1\)York (1972); Gibbons, Hawking (1977); Katz (1985); Hawking, Horowitz (1996); Katz, Bičák, Lynden-Bell (1997).
Examples of boundary terms:

1) The Gibbons-Hawking-York (GHY) BT,

\[ I_{GHY} = 2\kappa \int_{\partial M_4} d^3x \sqrt{-h} K, \]

is such that the variation of \( I_{Dir,GHY} \equiv I_{EH} + I_{GHY} \) is,

\[ \delta I_{Dir,GHY} = \kappa \int_{\partial M_4} d^3x \sqrt{|h|} b_{ij} \delta h^{ij} \]

with \( b_{ij} \equiv K_{ij} - h_{ij} K \). Here \( b_{ij} \) can be identified\(^2\) with the canonical momenta in Hamiltonian formalism.

\(^2\)As mentioned e.g. in R. Olea and O. Mišković [arXiv:0706.4460], this relation is useful to understand Israel junction conditions for branes as a discontinuity in the canonical momenta. For the EGB case see: S.C. Davis [arXiv: hep-th/0208205]; E. Gravanis and S. Willison, [arXiv: hep-th/0209076]
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1) The Gibbons-Hawking-York (GHY) BT,

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To obtain finite charges is still necessary to make a background substraction, which is made by Hawking-Howrowitz BT,

\[ I_{HH} = 2\kappa \int_{\partial M_4} d^3x \sqrt{-h} (K - \bar{K}), \]

\[ \text{As mentioned e.g. in R. Olea and O. Mišković [arXiv:0706.4460], this relation is useful to understand Israel junction conditions for branes as a discontinity in the canonical momenta. For the EGB case see: S.C. Davis [arXiv: hep-th/0208205]; E. Gravanis and S. Willison, [arXiv: hep-th/0209076]} \]
2) Noticing that $\sqrt{-g}R = \sqrt{-g}G + \partial_\mu (\sqrt{-g}v^\mu)$ with 

\[ G = g^{\mu\nu} \left( \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda} \right) \]

and 

\[ v^\mu = g^{\nu\rho} \Gamma^\mu_{\nu\rho} - g^{\mu\nu} \Gamma^\rho_{\nu\rho} \]

the simplest case would be to subtract the divergence $\partial_\mu (\sqrt{-g}v^\mu)$ from $I_{EH}$.

3) The Katz BT,

\[ I_K = \kappa \int_{\mathcal{M}_4} d^4x \partial_\mu (\sqrt{-g}k^\mu_K) \]

\[ k^\mu_K = - (g^{\nu\rho} \Delta^\mu_{\nu\rho} - g^{\mu\nu} \Delta^\rho_{\nu\rho}) , \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho} . \]
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I_K = \kappa \int_{\mathcal{M}_4} d^4 x \partial_\mu (\sqrt{-g}k^\mu_K) = 2\kappa \int_{\partial\mathcal{M}_4} d^3 x \sqrt{-h} \left( K - \frac{1}{2} (h^{ij} + \bar{h}^{ij}) K_{ij} \right),
\]

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k^\mu_K = - (g^{\nu\rho} \Delta^\mu_{\nu\rho} - g^{\mu\nu} \Delta^\rho_{\nu\rho}), \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho}.
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2) Noticing that $\sqrt{-g} R = \sqrt{-g} G + \partial_{\mu} (\sqrt{-g} v^{\mu})$ with 

$$G = g^{\mu \nu} \left( \Gamma_{\mu \rho}^{\lambda} \Gamma_{\nu \lambda}^{\rho} - \Gamma_{\mu \nu}^{\rho} \Gamma_{\rho \lambda}^{\lambda} \right)$$

and $v^{\mu} = g^{\nu \rho} \Gamma_{\nu \rho}^{\mu} - g^{\mu \nu} \Gamma_{\nu \rho}^{\rho}$ the simplest case would be to subtract the divergence $\partial_{\mu} (\sqrt{-g} v^{\mu})$ from $I_{EH}$.

3) The Katz BT,

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I_{K} = \kappa \int_{\mathcal{M}_{4}} d^{4}x \partial_{\mu} (\sqrt{-g} k_{K}^{\mu}) = 2\kappa \int_{\partial \mathcal{M}_{4}} d^{3}x \sqrt{-h} \left( K - \frac{1}{2} \left( h_{ij} + \bar{h}_{ij} \right) \bar{K}_{ij} \right),
\]

\[
k_{(1)}^{\mu} = - \left( g^{\nu \rho} \Delta_{v \rho}^{\mu} - g^{\mu \nu} \Delta_{v \nu}^{\rho} \right), \quad \Delta_{v \nu}^{\rho} = \Gamma_{v \nu}^{\rho} - \bar{\Gamma}_{v \nu}^{\rho}.
\]

allows to define $I_{Dir,K} \equiv I_{EH} + I_{K}$, whose variation

\[
\delta I_{Dir,K} = \kappa \int_{\partial \mathcal{M}_{4}} d^{3}x \sqrt{|h|} \left( b_{ij} \delta h^{ij} + c_{ij} \delta h^{ij} \right), \quad c_{ij} = \frac{1}{2} h_{ij} \left( h^{lk} + \bar{h}^{lk} \right) \bar{K}_{lk} - \bar{K}_{ij}
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2) Noticing that $\sqrt{-g}R = \sqrt{-g}G + \partial_\mu (\sqrt{-g}v^\mu)$ with 
\[ G = g^{\mu\nu} \left( \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda} \right) \]
and $v^\mu = g^{\nu\rho} \Gamma^\mu_{\nu\rho} - g^{\mu\nu} \Gamma^\rho_{\nu\rho}$ the simplest case would be to substract the divergence $\partial_\mu (\sqrt{-g}v^\mu)$ from $I_{\text{EH}}$.

3) The Katz BT,
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\]
\[
k^\mu_{(1)} = - \left( g^{\nu\rho} \Delta^\mu_{\nu\rho} - g^{\mu\nu} \Delta^\rho_{\nu\rho} \right), \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho}.
\]
allows to define $I_{\text{Dir}, K} \equiv I_{\text{EH}} + I_K$, whose variation
\[
\delta I_{\text{Dir}, K} = \kappa \int_{\partial \mathcal{M}_4} d^3 x \sqrt{|h|} \left( b_{ij} \delta h^{ij} + c_{ij} \delta h^{ij} \right), \quad c_{ij} = \frac{1}{2} h_{ij} \left( h^{lk} + \bar{h}^{lk} \right) \bar{K}_{lk} - \bar{K}_{ij}
\]
The regularized Dirichlet action, in the context of Katz, Bicak, Lynden-Bell (1997) approach, is defined as $I_{\text{reg}, KBL} \equiv I_{\text{EH}} + I_K - \bar{I}_{\text{EH}}$. 
On one hand, the GHY procedure was generalized Myers (1987) to the case of Lovelock gravity,

\[ I_L = \sum_{p=0}^{\left\lfloor \frac{D-1}{2} \right\rfloor} \alpha_p I^{(p)}, \quad I^{(p)} = \frac{(D - 2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x \delta_{\mu_1 \cdots \mu_2} R_{\nu_1 \nu_2} \cdots R_{\nu_{2p-1} \nu_{2p}}. \]

For example, the Gibbons-Hawking-Myers (GHM) BT for the Gauss-Bonnet action \( I^{(2)} \) is

\[ I^{(2)}_{\text{GHM}} = 4 (D - 4)! \int_{\mathcal{M}_D} d^d x \sqrt{-h} \delta_{i_1 j_2 i_3}^{i_1 j_2 i_3} K_{j_1} \left( \frac{1}{2} R_{j_2 j_3} (h) - \frac{1}{3} K_{j_2} K_{j_3} \right). \]
On one hand, the GHY procedure was generalized Myers (1987) to the case of Lovelock gravity,

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For example, the Gibbons-Hawking-Myers (GHM) BT for the Gauss-Bonnet action \( I^{(2)} \) is

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There are two issues that allows to understand this generalization:

- The GHY procedure can be written in differential forms
- In that language the GHY term can be understood as a **dimensional continuation** (DC) the BT appearing in the Euler theorem.
On the other hand, up to know, a generalization of the Katz vector to Lovelock gravity is not known in the literature. In 2003, N. Deruelle, J. Katz and S. Ogushi constructed the vector,

\[ k^\mu_{\text{DKO}} = 2k^{(2)}_\mu R + 8R^\beta_\alpha \Delta^\mu_\alpha - 8R^\mu_\alpha \Delta^\beta_\alpha - 4\Delta^{[\alpha}_\gamma R^\mu_\rho \Delta^{\beta\gamma}_\lambda \equiv g^{\rho\alpha} \Delta^\mu_\lambda , \]

in such a way that gives the mass of the Boulware-Deser bh in Einstein-Gauss-Bonnet (EGB) theory. However,

- this vector does not solve the Dirichlet problem
- there is too much freedom to construct a vector from \((g_{\mu\nu}, \Delta^\mu_\nu)\).

In particular, we have noticed that is possible to add in many ways terms of the following type

\[ k^\mu_{\text{DKO}} \to k^\mu_{\text{DKO}} + P_{\mu_1\mu_2\mu_3\mu_4}^{\nu_2\nu_3\nu_4} \Delta^\mu_1 \cdot \Delta^\mu_2 \cdot \Delta^\mu_3 \cdot \Delta^\mu_4 + Q_{\mu_1\mu_2\mu_3\mu_4}^{\nu_2\nu_3\nu_4} \Delta^\mu_1 \cdot \Delta^\mu_2 \cdot \Delta^\mu_3 \cdot \Delta^\mu_4 , \]

in such a way that the value of the mass is not affected.
On this work we will show that a formulation of the Katz procedure in the language of **differential forms** (missing up to now) plus topological arguments gives a guideline to construct the vector

\[
k_{(2)}^\mu = 2k_{(1)}^\mu R - 4k_\alpha R_\alpha^\mu - 8\Delta_\beta^{[\alpha \mu]} R_\alpha^\beta - 4\Delta_\gamma^{[\alpha \beta]} g_{\gamma \rho} R_\rho^\mu \\
+ k_{(1)}^\mu \nabla_\alpha k_\alpha + 4\Delta_\gamma^{[\alpha \beta]} \nabla_\alpha \Delta_\beta^{[\mu \gamma]} + 8\Delta_\beta^{[\alpha \beta]} \nabla_\gamma \Delta_\beta^{[\mu \gamma]} + 8\Delta_\alpha^{[\mu \gamma]} \nabla_\beta \Delta_\gamma^{[\alpha \beta]} \\
- \frac{2}{3} \left[-k_{(1)}^\mu \Delta_\gamma^{[\beta \gamma]} \Delta_\alpha^{\alpha} + 2\Delta_\gamma^{[\gamma \mu]} \Delta_\rho^{\rho} \Delta_\gamma^{[\alpha \beta]} + 2\Delta_\gamma^{[\alpha \rho]} \Delta_\alpha^{\mu} \Delta_\gamma^{[\beta \gamma]} \\
+ k_{(1)}^\gamma \left(\Delta_\gamma^{[\mu \rho]} \Delta_\alpha^{\alpha} + \Delta_\alpha^{\alpha} \Delta_\gamma^{[\mu \rho]} - \Delta_\gamma^{[\mu \rho]} \Delta_\alpha^{\alpha}\right) \\
+ 2\Delta_\alpha^{\gamma} \left(\Delta_\gamma^{[\mu \rho]} \Delta_\rho^{\rho} + \Delta_\rho^{\rho} \Delta_\gamma^{[\mu \rho]} + \Delta_\gamma^{[\mu \rho]} \Delta_\rho^{\rho} + \Delta_\gamma^{[\mu \rho]} \Delta_\rho^{\rho}\right)\right],
\]

or

\[
k_{(2)}^\mu = -\delta_{\mu
u_1\nu_2\nu_3\nu_4}^{\mu_1\mu_2\mu_3\mu_4} \Delta_\nu_2^{\mu_2} \left(R^{\mu_3\mu_4}_{\nu_3\nu_4} - \nabla_\nu_3 \Delta_\nu_4^{\mu_3\mu_4} + \frac{2}{3} g_{\gamma \rho} \Delta_\nu_3^{[\mu_3 \alpha]} \Delta_\nu_4^{[\beta \mu_4]}\right),
\]

which is **the one** allowing to solve the Dirichlet problem.

This will also give us a guideline to perform the *generalization* to a **generic Lovelock gravity**.
To make comparison with GHM procedure, in $D = 5$ with Gaussian coordinates we have shown that:

$$I^{(2)}_K = I^{(2)}_{GHM} + \int_{\partial \mathcal{M}_5} d^4x \sqrt{-h} E$$

where,

$$E = 4\delta^{i_1i_2i_3} K^{[i_1}_{i_2i_3} \left\{ -\frac{1}{2} \hat{\nabla}_{i_2} \Delta_{i_3} \right\} + K^{j_2}_{i_2} Z^{j_3}_{i_3} - \frac{1}{3} \left( K^{j_2}_{i_2} Z^{j_3}_{i_3} + Z^{j_2}_{i_2} \left( K^{j_3}_{i_3} + Z^{j_3}_{i_3} \right) \right) + \frac{1}{3} h_{i_2i_3} \Delta_{i_2} \Delta_{i_3} \right\}$$

$$\Delta_{i_1j_2} \left\{ -2\epsilon \hat{\nabla}_{i_2} K^{j_3}_{i_3} + \hat{\nabla}_{i_2} \left( K^{j_3}_{i_3} + Z^{j_3}_{i_3} \right) - \frac{2}{3} h_{i_2i_3} \left( K^{l_2}_{i_2} + Z^{l_2}_{i_2} \right) \right\}$$

$$\Delta_{ik} = \Gamma_{ik} - \bar{\Gamma}_{ik}, \quad \Delta_{i} = h^{ik} \Delta_{ik}, \quad Z_{i} = -\frac{1}{2} \left( h^{ij} + \bar{h}^{ij} \right) \bar{K}_{i}.$$

The variation of $I^{(2)}_{Dir,K} \equiv I^{(2)}_{Dir} + I^{(2)}_K$ has the form

$$\delta I^{(2)}_{Dir,K} = \int_{\partial \mathcal{M}_D} d^4x \sqrt{|h|} \left( B_{ij} \delta h^{ij} + C_{ij} \delta h^{ij} \right)$$
EH gravity in the vielbein formulation

Vielbein and spin connection: \((g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}) \rightarrow (e^A, \omega^A_B)\)

\[ e^A = e^A_\mu dx^\mu, \quad \omega^A_B = \omega^A_{\mu B} dx^\mu \]

The relation between both geometrical descriptions is,

\[ e^A_\mu e^B_\nu g_{\mu\nu} = \eta_{AB}, \quad \omega^A_{\mu B} = e^A_\alpha e^\gamma_B \Gamma^\alpha_{\mu \gamma} + e^A_\alpha \partial_\mu e^\alpha_B. \]

The curvature and torsion two-forms are defined by,

\[ R^A_B = \frac{1}{2} R^A_{\mu \nu B} dx^\mu dx^\nu = d\omega^A_B + \omega^A_C \omega^C_B, \quad R^A_{\mu \nu B} = e^A_\alpha e^B_\beta R^\alpha_{\mu \nu}; \]

\[ T^A = \frac{1}{2} T^A_{\mu \nu} dx^\mu dx^\nu = de^A + \omega^A_B e^B, \quad T^A_{\mu \nu} = e^A_\lambda T^\lambda_{\mu \nu}. \]
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\]

To formulate GR, one needs to assume \(T^A = 0\) (i.e., \(\omega^A_{\mu B} = \omega^A_{\mu B} (e, \partial e)\)) and introduce the Levi-Civita symbols \(\varepsilon_{ABCD}, \varepsilon_{\mu\nu\alpha\beta}\).
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To formulate GR, one needs to assume \( T^A = 0 \) (i.e., \( \omega^A_B = \omega^A_{\mu B} (e, \partial e) \)) and introduce the Levi-Civita symbols \( \varepsilon_{ABCD}, \varepsilon_{\mu \nu \alpha \beta} \). Using their properties (like, e.g., \( \varepsilon_{ABCD} e^A_{\mu} e^B_{\nu} e^C_{\alpha} e^D_{\beta} = \sqrt{-g} \varepsilon_{\mu \nu \alpha \beta} \)) one gets,
EH gravity in the vielbein formulation

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To formulate GR, one needs to assume \(T^A = 0\) (i.e., \(\omega^A_{\mu B} = \omega^A_{\mu B} (e, \delta e)\)) and introduce the Levi-Civita symbols \(\epsilon_{ABCD}, \epsilon_{\mu\nu\alpha\beta}\). Using their properties (like, e.g., \(\epsilon_{ABCD} e^A_\mu e^B_\nu e^C_\alpha e^D_\beta = \sqrt{-g} \epsilon_{\mu\nu\alpha\beta}\)) one gets,

\[
I_{EH} [g] = \kappa \int_{\mathcal{M}_4} d^4x \sqrt{-g} R = \frac{\kappa}{2} \int_{\mathcal{M}_4} \epsilon_{ABCD} R^{AB} e^C e^D \equiv I_{EH} [e],
\]
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To formulate GR, one needs to assume \(T^A = 0\) (i.e., \(\omega^A_{\mu B} = 0\)) and introduce the Levi-Civita symbols \(\varepsilon_{ABCD}, \varepsilon_{\mu \nu \alpha \beta}\). Using their properties (like, e.g., \(\varepsilon_{ABCD} e^A_\mu e^B_\nu e^C_\alpha e^D_\beta = \sqrt{-g} \varepsilon_{\mu \nu \alpha \beta}\)) one gets,

\[ I_{EH} [g] = \kappa \int_{\mathcal{M}} d^4 x \sqrt{-g} R = \frac{\kappa}{2} \int_{\mathcal{M}} \varepsilon_{ABCD} R^{AB} e^C e^D \equiv I_{EH} [e], \]

\[ T^A \rightarrow 0 \quad \delta I_{EH} = \frac{\kappa}{2} \int_{\partial \mathcal{M}} d (\varepsilon_{ABCD} \delta \omega^{AB} e^C e^D). \]
**Euler theorem and the GHY term**

The *Euler theorem for $\mathcal{M}_2$*

$$\int_{\mathcal{M}_2} \sqrt{-g} R d^2 x = -4\pi \chi (\mathcal{M}_2)$$
Euler theorem and the GHY term

The Euler theorem for $\mathcal{M}_2$ with $\partial \mathcal{M}_2 \neq 0$ and $ds^2 = N^2 (r) \, dr^2 - h(r,t) \, dt^2$ reads

$$\int_{\mathcal{M}_2} \sqrt{-g} R d^2 x = -4\pi \chi (\mathcal{M}_2) + \int_{\partial \mathcal{M}_2} \sqrt{-h} K dt, \quad K = \frac{1}{2N} \partial_r h$$
Euler theorem and the GHY term

The Euler theorem for $M_2$ with $\partial M_2 \neq 0$ and $ds^2 = N^2 (r) \, dr^2 - h (r, t) \, dt^2$ reads

$$\int_{\mathcal{M}_2} \sqrt{-g} R \, dx = -4\pi \chi (\mathcal{M}_2) + \int_{\partial \mathcal{M}_2} \sqrt{-h} K \, dt, \quad K = \frac{1}{2N} \partial_r h$$

To write this in differential forms it is necessary to introduce a reference manifold $\mathcal{\bar{M}}_2$ with metric $\bar{ds}^2 = N^2 (r) \, dr^2 - \bar{h} (t) \, dt^2$, such that $\partial M_2 = \partial \mathcal{\bar{M}}_2$. Then,

$$\int_{\mathcal{M}_2} \varepsilon_{AB} R^{AB} = -4\pi \chi (\mathcal{M}_2) + \int_{\partial \mathcal{M}_2} \varepsilon_{AB} \theta^{AB},$$

where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ and $\bar{\omega}^{AB}$ is the reference spin connection.


**Euler theorem and the GHY term**

The *Euler theorem* for $\mathcal{M}_2$ with $\partial \mathcal{M}_2 \neq 0$ and $ds^2 = N^2 (r) \, dr^2 - h (r, t) \, dt^2$ reads

$$\int_{\mathcal{M}_2} \sqrt{-g} \, R \, d^2 x = -4 \pi \chi (\mathcal{M}_2) + \int_{\partial \mathcal{M}_2} \sqrt{-h} \, K \, dt, \quad K = \frac{1}{2N} \partial_r h$$

To write this in differential forms it is necessary to introduce a reference manifold $\tilde{\mathcal{M}}_2$ with metric $d\bar{s}^2 = N^2 (r) \, dr^2 - \bar{h} (t) \, dt^2$, such that $\partial \mathcal{M}_2 = \partial \tilde{\mathcal{M}}_2$. Then,

$$\int_{\mathcal{M}_2} \varepsilon_{AB} R^{AB} = -4 \pi \chi (\mathcal{M}_2) + \int_{\partial \mathcal{M}_2} \varepsilon_{AB} \theta^{AB},$$

where $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ and $\bar{\omega}^{AB}$ is the reference spin connection. One can recover the tensorial Euler theorem using

$$\omega^{1a} = K^a = e^a_j K^j_i dx^i, \quad K_{ij} = \frac{1}{2N} \partial_r h_{ij}$$

$$\bar{\omega}^{1a} = \bar{K}^a = \bar{e}^a_j \bar{K}^j_i dx^i = 0, \quad \bar{K}_{ij} = 0$$

$$\theta^{1a} = K^a, \quad \theta^{ab} = \omega^{ab} \bigg|_{\partial \mathcal{M}} - \bar{\omega}^{ab} \bigg|_{\partial \mathcal{M}} = 0.$$

(1)

where the indices are such that $a = 0$, $x^i = t$. 
**Dimensional continuation (DC) procedure:**

\[
\varepsilon_{AB} R^{AB} \xrightarrow{\text{DC}} \left\{ \begin{array}{l}
I_{\text{EH}} = \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} R^{AB} e^C e^D = \kappa \int_{\partial M_4} d^4 x \sqrt{-g} R , \\
I_{\text{GHY}} = \frac{\kappa}{2} \int_{\partial M_4} \beta_1 = - \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \theta^{AB} e^C e^D \\
\quad = \kappa \int_{\partial M_4} \varepsilon_{abc} K^a e^c e^d = 2\kappa \int_{\partial M_4} d^3 x \sqrt{-h} K ,
\end{array} \right.
\]

where \( \theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \) with \( \omega^{AB} \) and \( \bar{\omega}^{AB} \) being the spin connections of

\[
ds^2 = N^2(\tau)\, d\tau^2 + h_{ij}(r,x)\, dx^i dx^j ; \quad \bar{d}s^2 = \bar{N}^2(\tau)\, d\tau^2 + \bar{h}_{ij}(x)\, dx^i dx^j .
\]
Dimensional continuation (DC) procedure:

\[ \varepsilon_{AB} R^{AB} \xrightarrow{DC} \left\{ I_{EH} = \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} R^{AB} \varepsilon^{C} e^{D} = \kappa \int_{\partial M_4} d^4 x \sqrt{-g} R, \right. \]

\[ \varepsilon_{AB} \theta^{AB} \xrightarrow{DC} \left\{ I_{GHY} = \frac{\kappa}{2} \int_{\partial M_4} \beta(1) = -\frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \theta^{AB} e^{C} e^{D} \right. \]

\[ = \kappa \int_{\partial M_4} \varepsilon_{abc} K^a e^c e^d = 2 \kappa \int_{\partial M_4} d^3 x \sqrt{-h} K, \]

where \( \theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \) with \( \omega^{AB} \) and \( \bar{\omega}^{AB} \) being the spin connections of

\[ ds^2 = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j; \quad d\bar{s}^2 = \bar{N}^2(r) dr^2 + \bar{h}_{ij}(x) dx^i dx^j. \]

Using

\[ \delta I_{EH} = \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \delta \omega^{AB} e^C e^D = -\kappa \int_{\partial M_4} \varepsilon_{abc} \delta K^a e^b e^c, \]
Dimensional continuation (DC) procedure:

\[
\varepsilon_{AB} R^{AB} \xrightarrow{\text{DC}} \left\{ I_{\text{EH}} = \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} R^{AB} e^C e^D = \kappa \int_{\partial M_4} d^4 x \sqrt{-g} R, \right. \\
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\]

where \( \theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \) with \( \omega^{AB} \) and \( \bar{\omega}^{AB} \) being the spin connections of

\[
ds^2 = N^2 (r) \, dr^2 + h_{ij} (r, x) \, dx^i dx^j \; ; \; \; d\bar{s}^2 = \bar{N}^2 (r) \, dr^2 + \bar{h}_{ij} (x) \, dx^i dx^j .
\]

Using

\[
\delta I_{\text{EH}} = \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \delta \omega^{AB} e^C e^D = -\kappa \int_{\partial M_4} \varepsilon_{abc} \delta K^a e^b e^c , \\
\implies \delta I_{\text{Dir,GHY}} = -\kappa \int_{\partial M_4} \varepsilon_{abc} \delta K^a e^b e^c + \kappa \int_{\partial M_4} \varepsilon_{abc} \left( \delta K^a e^b e^c + 2K^a e^b \delta e^c \right)
\]

\[
= 2\kappa \int_{\partial M_4} \varepsilon_{abc} K^a e^b \delta e^c = \kappa \int_{\partial M_4} d^3 x \sqrt{|h|} b_{ij} \delta h^{ij} .
\]
Similar constructions can be done in Lovelock gravity in $D = d + 1$ dimensions,

$$I_L = \left[\frac{D-1}{2}\right] \sum_{p=0}^{\infty} \alpha_p I^{(p)},$$

$$I^{(p)} = \frac{(D-2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x \delta_{\mu_1...\mu_{2p}}^{\nu_1...\nu_{2p}} R^{\mu_1\mu_2} \cdots R^{\mu_{2p-1}\mu_{2p}}_{\nu_1...\nu_{2p}}$$

$$= \int_{\mathcal{M}_D} \varepsilon_{A_1...A_D} R^{A_1A_2} \cdots R^{A_{2p-1}A_{2p}} e^{A_{2p+1}} \cdots e^{A_D}.$$

In this case, the previous procedure to solve the Dirichlet problem was generalized by Myers (1987):

The so called Gibbons-Hawking-Myers (GHM) term $\beta^{(p)}$, which solves Dirichlet problem for each $I^{(p)}$, is constructed as the DC of the BT appearing in the $2p$-dimensional Euler theorem.
For example, the GHM term for the Gauss-Bonnet case,

\[ I^{(2)} = (D - 2)! \int_{\mathcal{M}_D} d^D x \sqrt{-g} \left( R^2 - 4R^{\mu \nu} R_{\mu \nu} + R^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} \right) \]

\[ = \int_{\mathcal{M}_D} \varepsilon_{A_1 \ldots A_D} R^{A_1 A_2} R^{A_3 A_4} e^{A_5} \ldots e^{A_D}, \]

\[ \delta I^{(2)} = 2 \int_{\mathcal{M}_D} \varepsilon_{A_1 \ldots A_D} \delta \omega^{A_1 A_2} R^{A_3 A_4} e^{A_5} \ldots e^{A_D}, \]

is given by,

\[ \beta^{(2)} = -2\varepsilon_{A_1 \ldots A_D} \theta^{A_1 A_2} \left( R^{A_3 A_4} (h) + \frac{1}{3} \theta^{A_3} \theta^{BA_4} \right) e^{A_5} \ldots e^{A_D} \]

\[ = 4 (D - 4)! d^d x \sqrt{-h} \delta^{j_1 j_2 j_3} k_i^{i_1} \left( \frac{1}{2} \mathcal{R}^{i_2 i_3}_{j_2 j_3} (h) - \frac{1}{3} k_i^{j_2} k_i^{j_3} \right), \]

And this term can be regarded as the dimensional continuation of the \textit{BT} in the 4-dimensional Euler theorem,

\[ \int_{\mathcal{M}_4} \varepsilon_{ABCD} R^{AB} R^{CD} = 2 (4\pi)^2 \chi (\mathcal{M}_4) + 2 \int_{\partial \mathcal{M}_4} \varepsilon_{ABCD} \theta^{AB} \left( \mathcal{R}^{CD} (h) + \frac{1}{3} \theta^C \theta^{ED} \right) e^C e^D, \]
Katz procedure in vielbein formalism

The Katz BT can be written as,

\[ I_K = \kappa \int_{\partial \mathcal{M}_D} d^D x \partial_{\mu} \left( \sqrt{-g} k_{K}^\mu \right), \]

where,

\[ k_{K}^\mu = - \left( g^{\nu\rho} \Delta_{v\rho}^\mu - g^{\mu\nu} \Delta_{v\rho}^\rho \right), \]
\[ \Delta_{v\rho}^\mu = \Gamma_{v\rho}^\mu - \bar{\Gamma}_{v\rho}^\mu, \]

A smooth mapping \( \sigma : \mathcal{M} \rightarrow \bar{\mathcal{M}} \) is chosen such that the same coordinates \( x^\mu \) are used for the points \( P \in \mathcal{M} \) and \( \bar{P} = \sigma (P) \in \bar{\mathcal{M}} \). As a consequence:

- quantities like \( g_{\mu\nu} + \bar{g}_{\mu\nu} \) transforms as a tensor,
- \( \Delta_{v\rho}^\mu = \Gamma_{v\rho}^\mu - \bar{\Gamma}_{v\rho}^\mu \) is a well defined tensor too.
Katz procedure in vielbein formalism

We can introduce vielbeins in both manifolds,

<table>
<thead>
<tr>
<th>Manifold $M$</th>
<th>Manifold $\tilde{M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall P \in M, \exists x^\mu = x^\mu (y^A)$ such that, $e^\mu_A e^\nu_B \delta_{\mu\nu} = \eta_{AB}, \text{ with } e^\mu_A (P) = \frac{\partial x^\mu}{\partial y^A} (P)$.</td>
<td>$\forall \tilde{P} = \sigma (P) \in \tilde{M} \exists x^\mu = x^\mu (\tilde{y}^A)$ such that, $\tilde{e}^\mu_A \tilde{e}^\nu_B \delta_{\mu\nu} = \eta_{AB}, \text{ with } \tilde{e}^\mu_A (\tilde{P}) = \frac{\partial x^\mu}{\partial \tilde{y}^A} (\tilde{P})$.</td>
</tr>
</tbody>
</table>

\[
\tilde{\omega}^A_{\mu} = \tilde{e}^A_{\alpha} e^B \gamma \tilde{\Gamma}^\alpha_{\mu\gamma} + \tilde{e}^A_{\alpha} \partial_{\mu} \tilde{e}^B_{\alpha}, \quad \text{(2)}
\]

\[
\omega^A_{\mu} = e^A_{\alpha} e^B \gamma \Gamma^\alpha_{\mu\gamma} + e^A_{\alpha} \partial_{\mu} e^B_{\alpha}. \quad \text{(3)}
\]

To solve the Dirichlet problem in the EH case one might naively try to construct the object

\[
\beta^{(1)} (\omega, \tilde{\omega}) = -\varepsilon_{A_1 \ldots A_D} \theta^{A_1 A_2} e^{A_3} \ldots e^{A_D},
\]

similar to the one constructed in the GHY case, but where now $\theta = \omega - \tilde{\omega}$ with $\tilde{\omega}$ being associated with the background metric, instead of a product metric.
However, this does not allow us to transform $\beta_{(1)}$ to the tensorial language. The reason is that the vielbeins are related as

$$ e^A_\mu = \Omega^A_B (x) \tilde{e}^B_\mu, $$

where $\Omega^A_B (x)$ is not a Lorentz rotation, i.e., $\Omega^T \eta \Omega \neq \eta$ and this leads

$$ \bar{\omega}^A_{\mu B} = \left( \Omega^{-1} \right)^A_C \Omega^D_B \epsilon^C e^D e^B \tilde{\Gamma}^A_{\mu \gamma} + \left( \Omega^{-1} \right)^A_C e^C \partial_\mu \left( e^D \Omega^D_B \right). $$

Then it is not possible to use the identity $\epsilon_{A_1 \ldots A_D} e^A_\mu_1 \cdots e^A_\mu_D = \sqrt{-g} \epsilon_\mu_1 \cdots \mu_D$, which is fundamental to make the translation to tensorial language.

Therefore, describing independently each manifold with differential forms is **not enough** to write the Katz divergence term in that language.
The new relevant geometrical object to introduce is the "hybrid" spin connection,

\[
\tilde{\omega}_{\mu}^{AB} = \tilde{e}_{\alpha}^{A} \tilde{e}_{\gamma}^{B} \tilde{\Gamma}_{\mu \gamma}^{\alpha} + \tilde{e}_{\alpha}^{A} \partial_{\mu} \tilde{e}_{\alpha}^{B} = e_{\alpha}^{A} e_{\gamma}^{B} \bar{\Gamma}_{\mu \gamma}^{\alpha} + e_{\alpha}^{A} \partial_{\mu} e_{\alpha}^{B} ,
\]

where \( \tilde{e}_{\alpha}^{A} \) and \( \tilde{\Gamma}_{\mu \gamma}^{\alpha} \) are the vielbein and the Christoffel connection of a another\(^3\) reference manifold \( \tilde{M}_{D} \). Then, defining

\[
I_{K} = \frac{\kappa}{(D-2)!} \int_{\partial M_{D}} \beta^{(1)}, \quad \beta^{(1)} = \varepsilon_{A_{1}...A_{D}} \theta^{A_{1}A_{2}} e^{A_{3}} \ldots e^{A_{D}}
\]

\[
\theta_{\mu}^{AB} \equiv \theta_{\mu}^{AB} (\omega, \tilde{\omega}) = \omega_{\mu}^{AB} - \tilde{\omega}_{\mu}^{AB} = e_{\alpha}^{A} e_{\gamma}^{B} (\Gamma_{\mu \gamma}^{\alpha} - \bar{\Gamma}_{\mu \gamma}^{\alpha}) = e_{\alpha}^{A} e_{\gamma}^{B} \Delta_{\mu \gamma}^{\alpha} ,
\]

we have shown that the following relation holds,

\[
\frac{1}{(D-2)!} d\beta^{(1)} = d^{D} x \partial_{\mu} \left( \sqrt{-g} k_{K}^{\mu} \right) ,
\]

i.e., the Katz BT can also be written in differential forms.

\(^3\)We also choose a smooth mapping allowing us to use the same coordinates \( x^{\mu} \) for each point \( P \in M_{D} \) and \( \tilde{P} \in \tilde{M}_{D} \).
We notice that $\tilde{\mathcal{M}}_D$ is not a new independent manifold, because $\tilde{\omega}^A{}_{B\mu} = \tilde{\omega}^A{}_{B\mu} (\Gamma, e)$. Given $\mathcal{M}_D$ and $\tilde{\mathcal{M}}_D$ it is always possible to determine the geometry of $\tilde{\mathcal{M}}_D$.

For example, for $\mathcal{M}_D$ and $\tilde{\mathcal{M}}_D$ being static spherically symmetric spacetimes,

$$ds^2 = -f^2 (r) \, dt^2 + \frac{1}{g^2 (r)} \, dr^2 + r^2 d\Omega^2_{D-2}, \quad d\tilde{s}^2 = -\tilde{f}^2 (r) \, dt^2 + \frac{1}{\tilde{g}^2 (r)} \, dr^2 + r^2 d\Omega^2_{D-2},$$

we obtain,

$$d\tilde{s}^2 = -\tilde{f}^2 (r) \, dt^2 + \frac{1}{\tilde{g}^2 (r)} \, dr^2 + r^2 d\Omega^2_{D-2},$$

$$\tilde{g} (r) = \frac{\tilde{g}^2 (r)}{g (r)}, \quad \tilde{f}' (r) = \frac{g^2 (r) f (r) \tilde{f}' (r)}{\tilde{g}^2 (r) \tilde{f} (r)},$$

$$\tilde{\Gamma}^t_\mu r = \frac{f (r)}{\tilde{f} (r) \tilde{g}^2 (r)} \Gamma^t_\mu r, \quad \tilde{\Gamma}^r_\mu i = \frac{\tilde{g}^2 (r)}{g^2 (r)} \tilde{\Gamma}^r_\mu i, \quad \tilde{\Gamma}_ik = \tilde{\Gamma}_ik = \Gamma^i_ik.$$

Although this spin connection has vanishing torsion, it does not satisfy the metricity condition, and thus, $\tilde{\Gamma}$ is not a Christoffel symbol.
This proposal solves the Dirichlet problem in diff. forms language.

For example, in $D = 4$ the variation of $I_K = \frac{\kappa}{2} \int_{\partial M_4} \beta^{(1)}$ is

$$\delta I_K = \frac{\kappa}{2} \int_{\partial M_4} \delta \left[ -\varepsilon_{ABCD} \theta^{AB} e^C e^D \right]$$

$$= -\frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \delta \omega^{AB} e^C e^D + \frac{\kappa}{2} \int_{\partial M_4} \varepsilon_{ABCD} \delta \tilde{\omega}^{AB} e^C e^D - \kappa \int_{\partial M_4} \varepsilon_{ABCD} \theta^{AB} e^C \delta e^D$$

where we have used $\theta^{AB} = \omega^{AB} - \tilde{\omega}^{AB}$. Thus, $\delta I_K$ does cancel the BT comming from the variation of the EH action,

$$\delta I_{EH} = \frac{\kappa}{2} \int_{M_4} \varepsilon_{A_1 \ldots A_D} \delta \omega^{A_1 A_2 e^3 \ldots e^A_D}$$

and the rest vanishes for the Dirichlet condition $\delta e^A_{\mu} \big|_{\partial M_4} = 0$. Indeed,

$$d \left( \varepsilon_{ABCD} \delta \tilde{\omega}^{AB} e^C e^D \right) = d \left[ \varepsilon_{ABCD} f_E^{\nu, AB} \delta e^v e^C e^D - d \left( \varepsilon_{ABCD} e^A e^C e^D \right) \delta e^B \right] ,$$

$$f_E^{\nu, AB} (e, \partial e, \tilde{\Gamma}) = \left( \delta^{A}_E \delta^v e^B \gamma^{\alpha}_{\mu \gamma} + \delta^{B}_E \delta^v e^A \delta^{\beta}_{\alpha} \gamma^{\alpha}_{\mu \gamma} + \delta^{A}_E \delta^v \partial_{\mu} e^{B \alpha} \right) dx^\mu .$$
This proposal **solves** the Dirichlet problem in diff. forms language.

For example, in $D = 4$ the variation of $I_K = \frac{\kappa}{2} \int_{\partial M^4} \beta^{(1)}$ is

\[
\delta I_K = \frac{\kappa}{2} \int_{\partial M^4} \delta \left[ -\epsilon_{ABCD} \theta^{AB} e^C e^D \right] = -\frac{\kappa}{2} \int_{\partial M^4} \epsilon_{ABCD} \delta \omega^{AB} e^C e^D + \frac{\kappa}{2} \int_{\partial M^4} \epsilon_{ABCD} \delta \tilde{\omega}^{AB} e^C e^D - \kappa \int_{\partial M^4} \epsilon_{ABCD} \theta^{AB} e^C \delta e^D
\]

where we have used $\theta^{AB} = \omega^{AB} - \tilde{\omega}^{AB}$. Thus, $\delta I_K$ does cancel the $BT$ comming from the variation of the EH action,

\[
\delta I_{EH} = \frac{\kappa}{2} \int_{M^4} \epsilon_{A_1 \ldots A_D} \delta \omega^{A_1 A_2} e^{A_3} \ldots e^{A_D}
\]

and the rest vanishes for the Dirichlet condition $\delta e^A_{\mu} \big|_{\partial M^4} = 0$. Indeed,

\[
d \left( \epsilon_{ABCD} \delta \tilde{\omega}^{AB} e^C e^D \right) = d \left[ \epsilon_{ABCD} f_E^{\nu,AB} \delta e^\nu e^C e^D - d \left( \epsilon_{ABCD} e^A e^C e^D \right) \delta e^B \right],
\]

\[
f_E^{\nu,AB} (e, \partial e, \tilde{\Gamma}) = \left( \delta^A_E \delta^\nu_\alpha e^B_\gamma \tilde{\Gamma}^\alpha_{\mu \gamma} + \delta^B_E \delta^\nu_\alpha e^A_\gamma \delta^\beta_\gamma \tilde{\Gamma}^\alpha_{\mu \gamma} + \delta^A_E \delta^\nu_\alpha \partial_\mu e^{B\alpha} \right) dx^\mu.
\]
To show consistency with tensorial formalism, the variation of
$I_{\text{Dir,K}} = I_{\text{EH}} + I_{\text{K}},$
\[
\delta I_{\text{Dir,K}} = \frac{\kappa}{2} \int_{\partial M} \epsilon_{ABCD} \delta \tilde{\omega}^{A} e^C e^D - \kappa \int_{\partial M} \epsilon_{ABCD} \theta^{AB} e^C \delta e^D
\]
can be written in Gauss coordinates and then, transformed to
tensorial language. This way, we get
\[
\delta I_{\text{Dir,K}} = -\frac{\epsilon \kappa}{2} \int_{\partial M_D} \epsilon_{abc} (\delta \tilde{\omega}^{a} - \delta \tilde{\omega}^{an}) e^b e^c - \kappa \int_{\partial M_D} \epsilon \epsilon_{abc} (2\epsilon K^a + (\tilde{\omega}^{n} - \tilde{\omega}^{an})) e^b \delta e^c
\]
\[
= 2\epsilon \kappa \int_{\partial M_D} d^3 x \sqrt{|h|} \delta h^{ij} \left(-\frac{1}{2} \bar{K}_{ij} + \frac{1}{4} h_{jl} \left(h^{lk} + \bar{h}^{lk}\right) \bar{K}_{ik}\right)
\]
\[
- \epsilon \kappa \int_{\partial M_D} d^3 x \sqrt{|h|} \left[K h_{ij} - K_{ij} - \frac{1}{2} h_{ij} \left(h^{kl} + \bar{h}^{kl}\right) \bar{K}_{kl} + \frac{1}{2} h_{jl} \left(h^{lk} + \bar{h}^{lk}\right) \bar{K}_{ik}\right] \delta h^{ij}
\]
\[
= \epsilon \kappa \int_{\partial M_4} d^3 x \sqrt{|h|} \left(b_{ij} \delta h^{ij} + c_{ij} \delta h^{ij}\right),
\]
where
\[
b_{ij} = K_{ij} - h_{ij} K, \quad c_{ij} = \frac{1}{2} h_{ij} \left(h^{lk} + \bar{h}^{lk}\right) \bar{K}_{lk} - \bar{K}_{ij}
\]
Katz-like vectors in Lovelock theory

The generalization for a Lovelock term $I^{(p)}$ requires to consider a more general version of the Euler theorem:

*The Chern-Weil theorem (CWth)*

Let $A = A^M_T Mdx^\mu$ and $\bar{A} = \bar{A}^M_T Mdx^\mu$ be two one-forms gauge connection and $F, \bar{F}$ the corresponding strenghth fields (e.g., $F = dA + AA$). Then,

$$
\langle F^p \rangle - \langle \bar{F}^p \rangle = dT^{(2p-1)} (A, \bar{A}) ,
$$

$$
T^{(2p-1)} (A, \bar{A}) = p \int_0^1 dt \left\langle \theta F_t^{p-1} \right\rangle , \quad \theta = A - \bar{A} ,
$$

where $F_t$ is the strenght field of the connection $A_t = \bar{A} + t (A - \bar{A})$ which interpolates between $\bar{A}$ and $A$. Here, $T^{(2p-1)} (A, \bar{A})$ is called *transgression form*. Its variation is given by,

$$
\delta T^{(2p-1)} (A, \bar{A}) = p \left\langle F_t^{p-1} \delta A \right\rangle - p \left\langle \bar{F}_t^{p-1} \delta \bar{A} \right\rangle - p (p - 1) d \int_0^1 dt \left\langle \theta F_t^{p-2} \delta A_t \right\rangle .
$$
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\]

Consider the case where \( T_M = J_{AB} \) are Lorentz generators, so that
\[ \epsilon_{A_1 \ldots A_{2p}} = \langle J_{A_1 A_2} \cdots J_{A_{2p-1} A_{2p}} \rangle \]

\[ A = \frac{1}{2} \omega^{AB} J_{AB}, \quad \bar{A} = \frac{1}{2} \bar{\omega}^{AB} J_{AB}, \quad \tilde{A} = \frac{1}{2} \tilde{\omega}^{AB} J_{AB}, \]

\[ F = \frac{1}{2} R^{AB} J_{AB}, \quad \bar{F} = \frac{1}{2} \bar{R}^{AB} J_{AB}, \quad \tilde{F} = \frac{1}{2} \tilde{R}^{AB} J_{AB}, \]

where \( \bar{\omega} \) and \( \tilde{\omega} \) correspond to the product metric and hybrid cases.
\[ \varepsilon_{A_1...A_{2p}} = \langle J_{A_1A_2} \cdots J_{A_{2p-1}A_{2p}} \rangle \]

\[
A = \frac{1}{2} \omega^{AB} J_{AB}, \quad \bar{A} = \frac{1}{2} \bar{\omega}^{AB} J_{AB}, \quad \breve{A} = \frac{1}{2} \breve{\omega}^{AB} J_{AB},
\]

\[
F = \frac{1}{2} R^{AB} J_{AB}, \quad \bar{F} = \frac{1}{2} \bar{R}^{AB} J_{AB}, \quad \breve{F} = \frac{1}{2} \breve{R}^{AB} J_{AB},
\]

where \( \bar{\omega} \) and \( \breve{\omega} \) correspond to the product metric and hybrid cases.

- For \((\omega, \bar{\omega})\) the CWth reproduces the 2\(p\)-dimensional Euler theorem with the BT \(T^{(2p-1)}(\omega, \bar{\omega})\). The GHM term is

\[
I_{\text{GHM}}^{(p)} = \frac{\alpha_p}{(D-2)!} \int_{\mathcal{M}_D} d\beta_{(p)} \quad \text{where} \quad \beta_{(p)} \quad \text{is the DC of} \quad T^{(2p-1)}(\omega, \bar{\omega}).
\]
\[ \varepsilon_{A_1 \ldots A_{2p}} = \left\langle J_{A_1A_2} \cdots J_{A_{2p-1}A_2} \right\rangle \]

\[ A = \frac{1}{2} \omega^{AB} J_{AB}, \quad \bar{A} = \frac{1}{2} \bar{\omega}^{AB} J_{AB}, \quad \breve{A} = \frac{1}{2} \tilde{\omega}^{AB} J_{AB}, \]

\[ F = \frac{1}{2} R^{AB} J_{AB}, \quad \bar{F} = \frac{1}{2} \bar{R}^{AB} J_{AB}, \quad \breve{F} = \frac{1}{2} \tilde{R}^{AB} J_{AB}, \]

where \( \bar{\omega} \) and \( \tilde{\omega} \) correspond to the product metric and hybrid cases.

- For \((\omega, \bar{\omega})\) the CWth reproduces the \(2p\)-dimensional Euler theorem with the BT \( T^{(2p-1)} (\omega, \bar{\omega}) \). The GHM term is
  \[ I_{\text{GHM}}^{(p)} = \frac{\alpha_p}{(D-2)!} \int_{M_D} d\beta_{(p)} \] where \( \beta_{(p)} \) is the DC of \( T^{(2p-1)} (\omega, \bar{\omega}) \).

- For \((\omega, \tilde{\omega})\) the CWth give us the transgression form
  \[ T^{(2p-1)} (\omega, \tilde{\omega}) = -p \int_0^1 dt \varepsilon_{A_1 \ldots A_{2p}} \theta^{A_1A_2} \left( R^{A_3A_4} + tD \theta^{A_3A_4} + t^2 \eta_{B_1C_1} \theta^{[A_3B_1]} \theta^{[C_1A_4]} \right) \times \ldots \]
  \[ \times \left( R^{A_{2p-1}A_{2p}} + tD \theta^{A_{2p-1}A_{2p}} + t^2 \eta_{B_pC_p} \theta^{[A_{2p-1}B_p]} \theta^{[C_pA_{2p}]} \right), \]
\[ \varepsilon_{A_1 \cdots A_{2p}} = \left< J_{A_1 A_2} \cdots J_{A_{2p-1} A_{2p}} \right> \]

\[ A = \frac{1}{2} \omega^{AB} J_{AB}, \quad A = \frac{1}{2} \tilde{\omega}^{AB} J_{AB}, \quad \tilde{A} = \frac{1}{2} \tilde{\omega}^{AB} J_{AB}, \]

\[ F = \frac{1}{2} R^{AB} J_{AB}, \quad \tilde{F} = \frac{1}{2} \tilde{R}^{AB} J_{AB}, \quad \tilde{F} = \frac{1}{2} \tilde{R}^{AB} J_{AB}, \]

where \( \tilde{\omega} \) and \( \tilde{\omega} \) correspond to the product metric and hybrid cases.

- For \((\omega, \tilde{\omega})\) the CWth reproduces the \(2p\)-dimensional Euler theorem with the BT \( T^{(2p-1)}(\omega, \tilde{\omega}) \). The GHM term is
  \[
  I^{(p)}_{\text{GHM}} = \frac{\alpha_p}{(D-2)!} \int_{\mathcal{M}_D} d\beta^{(p)} \quad \text{where} \quad \beta^{(p)} \text{ is the DC of } T^{(2p-1)}(\omega, \tilde{\omega}).
  \]

- For \((\omega, \tilde{\omega})\) the CWth give us the transgression form
  \[
  T^{(2p-1)}(\omega, \tilde{\omega}) = -p \int_0^1 dt \varepsilon_{A_1 \cdots A_{2p}} \theta^{A_1 A_2} \left( R^{A_3 A_4} + tD \theta^{A_3 A_4} + t^2 \eta_{B_1 C_1} \theta^{[A_3 B_1]} \theta^{[C_1 A_4]} \right) \times \cdots \times \left( R^{A_{2p-1} A_{2p}} + tD \theta^{A_{2p-1} A_{2p}} + t^2 \eta_{B_p C_p} \theta^{[A_{2p-1} B_p]} \theta^{[C_p A_{2p}]} \right),
  \]

and we have shown the Katz-like boundary term is
  \[
  I^{(p)}_{\text{K}} = \frac{\alpha_p}{(D-2)!} \int_{\mathcal{M}_D} d\beta^{(p)} \quad \text{with} \quad \beta^{(p)} \text{ being the DC of } T^{(2p-1)}(\omega, \tilde{\omega}).
  \]
In particular, for $p = 2$,

$$T^{(3)}(\omega, \tilde{\omega}) = 2\varepsilon_{ABCD}\theta^{AB} \left( R^{CD} - \frac{1}{2} D\theta^{CD} + \frac{1}{3} \eta_{EF} \theta^{[CE]} \theta^{[FD]} \right)$$

Then we define the DC of $T^{(3)}(\omega, \tilde{\omega})$ as,

$$\beta^{(2)} = -2\varepsilon_{A_1 \ldots A_D} \theta^{A_1 A_2} \left( R^{A_3 A_4} - \frac{1}{2} D^{(W)} \theta^{A_3 A_4} + \frac{1}{3} \eta_{EF} \theta^{[A_3 E]} \theta^{[F A_4]} \right) e^{A_5} \ldots e^{A_D}.$$  

We have shown that $I_K^{(2)} = \frac{\alpha_2}{(D-2)!} \int_{\mathcal{M}_D} d\beta^{(2)}$ solves the Dirichlet problem and that the following relation holds

$$\frac{1}{(D-4)!} d\beta^{(2)} = \partial_\mu \left( \sqrt{-g} k^{(2)}_\mu \right) d^D x,$$

$$k^{(2)}_\mu = -\delta^{\mu v_2 \mu v_3 \mu v_4} \Delta^{\mu_1 \mu_2}_{v_2} \left( R^{\mu_3 \mu_4}_{v_3 v_4} - \nabla_{v_3} \Delta^{\mu_3 \mu_4}_{v_4} + \frac{2}{3} \eta_ab \Delta^{[\mu_3 a]}_{v_3} \Delta^{[\mu_4 b]}_{v_4} \right),$$

i.e., the term $\beta^{(2)}$, constructed from the CWth, give us the form that the Katz-like vector for the GB term must have.
Therefore, in the KLB approach, the regularized Dirichlet Lovelock action we are proposing is: \( I_{\text{L,reg-KBL}} = I_L + I_K - \bar{I}_L \), where

\[
I_L = \sum_{p=0}^{[D/2]} \alpha_p I^{(p)},
\]

\[
I^{(p)} = \int_{\mathcal{M}_D} \varepsilon_{A_1 \ldots A_D} R^{A_1 A_2} \ldots R^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \ldots e^{A_D}
\]

\[
= \frac{(D - 2p)!}{2^p} \int_{\mathcal{M}_D} \sqrt{-g} d^D x \, \delta^{v_1 \ldots v_{2p}}_{\mu_1 \ldots \mu_{2p}} \, R^{\mu_1 \mu_2} \ldots R^{\mu_{2p-1} \mu_{2p}},
\]

\[
I_K = \int_{\mathcal{M}_D} \sum_{p=0}^{[D/2]} \frac{\alpha_p}{(D - 2p)!} d\beta^{(p)} = \int_{\mathcal{M}_D} \sum_{p=0}^{[D/2]} \alpha_p \partial_\mu \left( \sqrt{-g} k^\mu_{(p)} \right) d^D x,
\]

\[
\beta^{(p)} = -p \int_0^1 dt \varepsilon_{A_1 \ldots A_D} \theta^{A_1 A_2} \left( R^{A_3 A_4} + t D \theta^{A_3 A_4} + t^2 \eta_{B_1 C_1} \theta^{[A_3 B_1]} \theta^{[C_1 A_4]} \right) \times \ldots
\]

\[
\quad \times \left( R^{A_{2p-1} A_{2p}} + t D \theta^{A_{2p-1} A_{2p}} + t^2 \eta_{B_{p-1} C_{p}} \theta^{[A_{2p-1} B_p]} \theta^{[C_p A_{2p}]} \right) e^{A_{2p+1}} \ldots e^{A_D},
\]

\[
k^\mu_{(p)} = -p \int_0^1 dt \delta^{\mu_2 \ldots \mu_{2p}}_{v_1 \ldots v_{2p}} \Delta_{\mu_2}^{v_1 v_2} \left( \frac{1}{2} R_{\mu_3 \mu_4}^{v_3 v_4} + t \nabla_{\mu_3} \Delta_{\mu_4}^{v_3 v_4} + t^2 g_{\alpha_1 \beta_1} \Delta_{\mu_3}^{v_3 \alpha_1} \Delta_{\mu_4}^{\beta_1 v_4} \right) \times
\]

\[
\quad \times \left( \frac{1}{2} R_{\mu_{2p-1} \mu_{2p}}^{v_{2p-1} v_{2p}} + t \nabla_{\mu_{2p-1}} \Delta_{\mu_{2p}}^{v_{2p-1} v_{2p}} + t^2 g_{\alpha_p \beta_p} \Delta_{\mu_{2p-1}}^{v_{2p-1} \alpha_p} \Delta_{\mu_{2p}}^{\beta_p v_{2p}} \right).
\]
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Conclusions

- The missing formulation in differential forms of the KLB procedure has been given.
- In particular, we have shown that there is a unique generalization of Katz vector for EGB and Lovelock gravities.
- The new BTs we have obtained are different from GHM BT.
- In the case of EGB action our vector is not DKO vector, but it gives the same mass for the Boulware-Deser bh.
Future directions

- It remains to check if it also gives the correct conserved charges for other cases like, e.g., in Pure Lovelock gravity\(^4\).
- We also want to compare our results with other procedures (the intrinsic and extrinsic regularization).
- In particular it would be interesting to find a relation with the A. Petrov approach.
- Then, it would be interesting to find an explicit proof that this proposal gives the correct conserved charges for families of solutions.
- To start, we will see if is possible to give a proof for locally asymptotically AdS solutions in Lovelock gravities with AdS vacuum (using Fefferman-Graham coordinates).

Future directions

- On the other hand, a recent application of the Katz procedure was made in the literature\(^5\) for EH gravity coupled with scalar and Maxwell fields. Thus, it would be interesting to investigate also the consequences of adding extra fields.

- Finally, it would also be interesting to investigate the consequences of these results with a possible generalization of the Israel Junction conditions for the Lovelock case.\(^6\).

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\(^5\) A. Anabalón, N. Deruelle and F. Julié [arXiv:1606.05849].

Thanks